

# ON THE NUMBER OF INTEGERS IN A GENERALIZED MULTIPLICATION TABLE

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**ABSTRACT.** Motivated by the Erdős multiplication table problem we study the following question: Given numbers  $N_1, \dots, N_{k+1}$ , how many distinct products of the form  $n_1 \cdots n_{k+1}$  with  $1 \leq n_i \leq N_i$  for  $i \in \{1, \dots, k+1\}$  are there? Call  $A_{k+1}(N_1, \dots, N_{k+1})$  the quantity in question. Ford established the order of magnitude of  $A_2(N_1, N_2)$  and the author of  $A_{k+1}(N, \dots, N)$  for all  $k \geq 2$ . In the present paper we generalize these results by establishing the order of magnitude of  $A_{k+1}(N_1, \dots, N_{k+1})$  for arbitrary choices of  $N_1, \dots, N_{k+1}$  when  $2 \leq k \leq 5$ . Moreover, we obtain a partial answer to our question when  $k \geq 6$ . Lastly, we develop a heuristic argument which explains why the limitation of our method is  $k = 5$  in general and we suggest ways of improving the results of this paper.

## CONTENTS

1. Introduction	2
1.1. The Erdős multiplication table problem and its generalizations	2
1.2. Main results	4
1.3. Outline of the paper	6
1.4. Notation	7
2. Heuristic arguments	7
2.1. Basic set-up and development of the main argument	8
2.2. Further analysis and optimality of condition (1.1)	12
3. Local-to-global estimates	15
3.1. Auxiliary results	15
3.2. The lower bound in Theorem 1.7	16
3.3. The upper bound in Theorem 1.7	18
3.4. Proof of Theorem 1.1	26
4. Linear constraints on a Poisson distribution	27
5. The upper bound in Theorem 1.5	32
5.1. Outline of the proof	32
5.2. Completion of the proof	34
6. The lower bound in Theorem 1.5: outline of the proof	40
7. The method of low moments	44
7.1. Interpolating between $L^1$ and $L^2$ estimates	44

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7.2. Combinatorial arguments	48
7.3. Proof of Lemma 6.2	51
8. The lower bound in Theorem 1.5: completion of the proof	51
8.1. Preliminaries	51
8.2. Estimates from order statistics	53
8.3. Proof of Lemmas 6.3 and 6.4	61
References	64

## 1. INTRODUCTION

**1.1. The Erdős multiplication table problem and its generalizations.** In 1955 Erdős posed the so-called *multiplication table problem* [E55]: Given a large number  $N$ , how many integers can be written as a product  $ab$  with  $a \leq N$  and  $b \leq N$ ? Erdős gave the first estimates of this quantity [E55, E60], which were subsequently sharpened by Tenenbaum [T84]. The problem of establishing the order of magnitude of the size of the  $N \times N$  multiplication table was completely solved by Ford in [Fo08a, Fo08b], where he showed that

$$A_2(N) := |\{ab : a \leq N \text{ and } b \leq N\}| \asymp \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log \log N)^{3/2}} \quad (N \geq 3),$$

where

$$Q(u) := \int_1^u \log t \, dt = u \log u - u + 1 \quad (u > 0).$$

More generally, we may ask the same question about higher dimensional analogues of the multiplication table problem, that is to say, we may ask for estimates of

$$A_{k+1}(N) := |\{n_1 \cdots n_{k+1} : n_i \leq N \ (1 \leq i \leq k+1)\}|.$$

In [K10a] the author determined the order of  $A_{k+1}(N)$  for every fixed  $k \geq 2$ : we have that

$$A_{k+1}(N) \asymp_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))} (\log \log N)^{3/2}} \quad (N \geq 3).$$

In the present paper we broaden our scope and study the number of integers that appear in a  $(k+1)$ -dimensional multiplication table when the side lengths of the table are different. More precisely, given numbers  $N_1, \dots, N_{k+1}$ , we seek uniform bounds on

$$A_{k+1}(N_1, \dots, N_{k+1}) := |\{n_1 \cdots n_{k+1} : n_i \leq N_i \ (1 \leq i \leq k+1)\}|.$$

Instead of studying  $A_{k+1}(N_1, \dots, N_{k+1})$  directly, we focus on a closely related function: Given  $x \geq 1$ ,  $\mathbf{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$  and  $\mathbf{z} = (z_1, \dots, z_k) \in \mathbb{R}^k$ , define

$$H^{(k+1)}(x, \mathbf{y}, \mathbf{z}) = |\{n \leq x : \exists d_1 \cdots d_k | n \text{ such that } y_i < d_i \leq z_i \ (1 \leq i \leq k)\}|.$$

The following theorem establishes the expected quantitative relation between  $A_{k+1}(N_1, \dots, N_{k+1})$  and  $H^{(k+1)}(x, \mathbf{y}, \mathbf{z})$ ; its proof will be given in Subsection 3.4.

**Theorem 1.1.** *Let  $k \geq 1$  and  $3 \leq N_1 \leq N_2 \leq \dots \leq N_{k+1}$ . Then*

$$A_{k+1}(N_1, \dots, N_{k+1}) \asymp_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \left(\frac{N_1}{2}, \dots, \frac{N_k}{2}\right), (N_1, \dots, N_k)\right).$$

In light of the above theorem, it suffices to restrict ourselves to the study of  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$ , which is slightly easier technically. What is more, bounds on  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  have applications beyond the multiplication table problem (for example, see [Fo08b] for several such applications when  $k = 1$ ). Before we state the results of this paper, we summarize some already known estimates on  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  in the theorem below. Briefly, this theorem gives the order of magnitude of  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  when the numbers  $\log y_1, \dots, \log y_k$  are roughly of the same size. In particular, it establishes the order of magnitude  $H^{(2)}(x, y, 2y)$  for all  $y$  and  $x$ . For a proof of it we refer the reader to [Fo08a, Fo08b] and [K10a]; the first two papers handle the case  $k = 1$  and the latter the case  $k \geq 2$ .

**Theorem 1.2** (Ford [Fo08a, Fo08b], Koukoulopoulos [K10a]). *Let  $k \geq 1$ ,  $c \geq 1$  and  $\delta > 0$ . Consider numbers  $x \geq 3$  and  $3 \leq y_1 \leq \dots \leq y_k \leq y_1^c$  with  $2^{k+1}y_1 \cdots y_k \leq x/y_1^\delta$ . Then*

$$H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \asymp_{k,c,\delta} \frac{x}{(\log y_1)^{Q(k/\log(k+1))} (\log \log y_1)^{3/2}}.$$

In this manuscript we extend Theorem 1.2 to a broader range of the parameters  $y_1, \dots, y_k$ . In particular, when  $2 \leq k \leq 5$  we establish the order of  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  for any choice of the parameters  $y_1, \dots, y_k$ . In order to state our results we introduce some notation. Given numbers  $3 = y_0 \leq y_1 \leq \dots \leq y_k$ , set

$$\ell_i = \log \frac{3 \log y_i}{\log y_{i-1}} \quad (1 \leq i \leq k).$$

Also, let  $i_1$  be the smallest element of  $\{1, \dots, k\}$  such that

$$\ell_{i_1} = \max\{\ell_i : 1 \leq i \leq k\}$$

and define  $\beta = \beta(k; \mathbf{y})$  by

$$\beta = \min \left\{ 1, \frac{(1 + \ell_1 + \dots + \ell_{i_1-1})(1 + \ell_{i_1+1} + \dots + \ell_k)}{\ell_{i_1}} \right\}.$$

Lastly, define  $\alpha = \alpha(k; \mathbf{y})$  implicitly, via the equation

$$\sum_{i=1}^k (k-i+2)^\alpha \log(k-i+2) \ell_i = \sum_{i=1}^k (k-i+1) \ell_i.$$

Note that

$$\alpha \geq \min_{1 \leq i \leq k} \frac{1}{\log(k-i+2)} \log \left( \frac{k-i+1}{\log(k-i+2)} \right) = \frac{1}{\log 2} \log \left( \frac{1}{\log 2} \right) = 0.528766373 \dots$$

as well as

$$\alpha \leq \max_{1 \leq i \leq k} \frac{1}{\log(k-i+2)} \log \left( \frac{k-i+1}{\log(k-i+2)} \right) = \frac{1}{\log(k+1)} \log \left( \frac{k}{\log(k+1)} \right) < 1$$

(here we used Lemma 2.2, which will be stated and proven in Section 2).

**Theorem 1.3.** *Let  $k \in \{2, 3, 4, 5\}$ ,  $x \geq 3$  and  $3 \leq y_1 \leq \dots \leq y_k$  be such that  $2^k y_1 \dots y_k \leq x/y_k$ . Then*

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \asymp \frac{\beta}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

As we shall see later, the hypothesis that  $k \in \{2, 3, 4, 5\}$  in the above theorem is necessary: when  $k \geq 6$  there are choices of the parameters  $y_1, \dots, y_k$  for which  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  has genuinely smaller order than what Theorem 1.3 predicts. However, if  $\log y_k$  is not much bigger than  $\log y_1$ , then the conclusion of Theorem 1.3 is valid. More precisely, we have the following result, which extends Theorem 1.2.

**Theorem 1.4.** *Let  $k \geq 6$ ,  $x \geq 3$  and  $3 \leq y_1 \leq \dots \leq y_k$  be such that  $2^k y_1 \dots y_k \leq x/y_k$  and  $\log y_k \leq (\log y_1)^{1+\delta}$  for a sufficiently small positive  $\delta = \delta(k)$ . Then*

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \asymp_k \frac{\log \frac{3 \log y_k}{\log y_1}}{(\log \log y_1)^{3/2}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

**1.2. Main results.** Both Theorems 1.3 and 1.4 are consequences of a more general estimate on  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$ , which is the main result of this paper.

**Theorem 1.5.** *Let  $k \geq 2$ ,  $x \geq 3$  and  $3 \leq y_1 \leq \dots \leq y_k$  be such that  $2^k y_1 \dots y_k \leq x/y_k$ . Then*

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \ll_k \frac{\beta}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

If we also assume that

$$(1.1) \quad \alpha \geq 1 + \epsilon - \frac{1}{\log(k+1)} \log \left( \frac{(k+1) \log(k+1) - 2 \log 2}{k-1} \right)$$

for some fixed  $\epsilon > 0$ , then

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \asymp_{k, \epsilon} \frac{\beta}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

Condition (1.1) is essentially optimal in the sense that for every fixed  $\gamma$  that satisfies

$$(1.2) \quad \frac{1}{\log 2} \log \left( \frac{1}{\log 2} \right) < \gamma < 1 - \frac{1}{\log(k+1)} \log \left( \frac{(k+1) \log(k+1) - 2 \log 2}{k-1} \right)$$

there is a choice of  $y_1 \leq \dots \leq y_k$  such that  $\alpha = \alpha(k; \mathbf{y}) = \gamma$  and for which the order of  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  is genuinely smaller than the one stated above. We shall discuss this further in the next section using some heuristic arguments. In relation to our comments following the statement of Theorem 1.3, note that the smallest value of  $k$  for which the range (1.2) is non-empty is  $k = 6$ .

Despite its optimality, condition (1.1) is not very easy to work with due to the implicit definition of  $\alpha$ . Below we state a weaker version of Theorem 1.5, whose hypotheses are easier to verify.

**Corollary 1.6.** *Let  $k \geq 2$ ,  $h \in \{1, \dots, k\}$ ,  $x \geq 3$  and  $3 \leq y_1 \leq \dots \leq y_k$  such that  $2^k y_1 \dots y_k \leq x/y_k$ ,*

$$\frac{1}{\log(k-h+2)} \log \left( \frac{k-h+1}{\log(k-h+2)} \right) > 1 - \frac{1}{\log(k+1)} \log \left( \frac{(k+1) \log(k+1) - 2 \log 2}{k-1} \right).$$

*and  $\log y_k \leq (\log y_h)^{1+\delta}$  for a sufficiently small positive  $\delta = \delta(k)$ . Then*

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \asymp_k \frac{\beta}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

*Proof.* We consider for the moment  $\delta$  to be a free parameter. Since  $\log y_k \leq (\log y_h)^{1+\delta}$ , we have that

$$\sum_{i=1}^h (k-i+2)^\alpha \log(k-i+2) \ell_i = \sum_{i=1}^h (1 + O_k(\delta))(k-i+1) \ell_i.$$

Therefore

$$\begin{aligned} \alpha &\geq \min_{1 \leq i \leq h} \frac{1}{\log(k-i+2)} \log \left( \frac{k-i+1}{\log(k-i+2)} \right) - O_k(\delta) \\ &= \frac{1}{\log(k-h+2)} \log \left( \frac{k-h+1}{\log(k-h+2)} \right) - O_k(\delta), \end{aligned}$$

by Lemma 2.2. So if we choose  $\delta$  small enough, then (1.1) holds and hence the desired result follows.  $\square$

Applying the above corollary with  $h = k \leq 5$  gives us Theorem 1.3 immediately. Similarly, Theorem 1.4 follows by Corollary 1.6 with  $h = 1$ ; we only need to check that

$$(1.3) \quad \frac{1}{\log(k+1)} \log \left( \frac{k}{\log(k+1)} \right) > 1 - \frac{1}{\log(k+1)} \log \left( \frac{(k+1) \log(k+1) - 2 \log 2}{k-1} \right)$$

or, equivalently, that

$$(k+1) \log(k+1) > k \log 4$$

for  $k \geq 2$ , which is indeed true.

The main tool we shall use in order to prove Theorems 1.1 and 1.5 is a result that reduces the counting in  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$ , which contains information about the local distribution of factorizations of integers, to the estimation of a certain sum which contains information about the global distribution of factorizations of integers. More precisely, for  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$  define

$$\mathcal{L}^{(k+1)}(\mathbf{a}) = \bigcup_{\substack{d_1 \dots d_i | a_1 \dots a_i \\ 1 \leq i \leq k}} [\log(d_1/2), \log d_1) \times \dots \times [\log(d_k/2), \log d_k),$$

and

$$L^{(k+1)}(\mathbf{a}) = \text{Vol}(\mathcal{L}^{(k+1)}(\mathbf{a})),$$

where “Vol” denotes the  $k$ -dimensional Lebesgue measure. Also, for  $1 \leq y < z$  set

$$\mathcal{P}_*(y, z) = \{n \in \mathbb{N} : \mu^2(n) = 1, p|n \Rightarrow y < p \leq z\}$$

and for  $\mathbf{t} = (t_1, \dots, t_k)$  with  $t_k \geq t_{k-1} \geq \dots \geq t_1 \geq 1 =: t_0$  set

$$\mathcal{P}_*^k(\mathbf{t}) = \{(a_1, \dots, a_k) \in \mathbb{N}^k : a_i \in \mathcal{P}_*(t_{i-1}, t_i) \ (1 \leq i \leq k)\}.$$

Then we have the following estimate.

**Theorem 1.7.** *Let  $k \geq 1$ ,  $x \geq 1$  and  $3 \leq y_1 \leq \dots \leq y_k$  with  $2^k y_1 \dots y_k \leq x/y_k$ . Then*

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \asymp_k \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-(k-i+2)} \sum_{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y})} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k}.$$

When  $k = 1$ , the above theorem is an immediate consequence of the results and the methods in [Fo08a]: see Lemmas 2.1 and 3.2 there. As an immediate consequence of Theorem 1.7, we have the following result.

**Corollary 1.8.** *Let  $k \geq 1$  and for  $i \in \{1, 2\}$  consider  $x_i \geq 1$  and  $\mathbf{y}_i = (y_{i,1}, \dots, y_{i,k}) \in [1, +\infty)^k$ . Assume that  $2^k y_{i,1} \dots y_{i,k} \leq x/y_{i,k}$  for  $i \in \{1, 2\}$  and that there exist constants  $c$  and  $C$  such that  $y_{1,j}^c \leq y_{2,j} \leq y_{1,j}^C$  for all  $j \in \{1, \dots, k\}$ . Then*

$$\frac{H^{(k+1)}(x_1, \mathbf{y}_1, 2\mathbf{y}_1)}{x_1} \asymp_{k,c,C} \frac{H^{(k+1)}(x_2, \mathbf{y}_2, 2\mathbf{y}_2)}{x_2}.$$

*Proof.* The result follows by Theorem 1.7, Lemma 2.1(a) and the standard estimate

$$(1.4) \quad \sum_{a \in \mathcal{P}_*(t, t^B)} \frac{\tau_m(a)}{a} \asymp_{m,B} 1 \quad (t \geq 1),$$

where

$$\tau_m(a) = \sum_{d_1 \dots d_{m-1} | a} 1 = \sum_{d_1 \dots d_m = a} 1 \quad (m \in \mathbb{N}, a \in \mathbb{N}).$$

□

When  $k = 1$ , a stronger version of the above corollary is known to be true: see Corollary 1 in [Fo08b].

**1.3. Outline of the paper.** The paper is organized in the following way: In Section 2 we demonstrate a heuristic argument in support of Theorem 1.5 and the optimality of condition (1.1). The first three subsections of Section 3 are devoted to establishing Theorem 1.7, whereas in the last one we prove Theorem 1.1. In Section 4 we develop some estimates related to the probability that a multidimensional Poisson random variable lies close to a hyperplane. Such estimates play a crucial role in the proof of Theorem 1.5. Also, in combination with the heuristic arguments of Section 2, they explain how the parameter  $\alpha$  makes its appearance in the statements of our results. In Section 5 we give the proof of the upper bound in Theorem 1.5. The main steps of the proof are described in Subsection 5.1 and proven in Subsection 5.2. The proof of the lower bound in Theorem 1.5 is divided in three sections: in Section 6 we describe the main steps of our argument. The first major such step is then carried out in Section 7. Finally, Section 8 contains the last piece of our argument and completes the proof of Theorem 1.5.

**1.4. Notation.** We make use of some standard notation. For  $n \in \mathbb{N}$  we use  $P^+(n)$  and  $P^-(n)$  to denote the largest and smallest prime factor of  $n$ , respectively, with the notational conventions that  $P^+(1) = 1$  and  $P^-(1) = +\infty$ . Also,  $\omega(n)$  denotes the number of distinct prime factors of  $n$ . Constants implied by  $\ll$ ,  $\gg$  and  $\asymp$  are absolute unless otherwise specified, e.g. by a subscript. Also, we use the letters  $c$  and  $C$  to denote constants, not necessarily the same ones in every place, possibly depending on certain parameters that will be specified by subscripts and other means. Also, bold letters always denote vectors whose coordinates are indexed by the same letter with subscripts, e.g.  $\mathbf{a} = (a_1, \dots, a_k)$  and  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$ . The dimension of the vectors will not be explicitly specified if it is clear by the context. Finally, we give a table of some basic non-standard notation that we will be using with references to page numbers for its definition.

Symbol	Page
$Q(u)$	2
$\alpha$	3
$\alpha_i$	8
$\beta$	3
$i_1$	3
$i_0$	8
$\ell_i$	3
$v_i$	33
$\Delta_r$	33
$\rho_m$	33
$\mathbf{e}_k, e_{k,i}$	19
$\tau_m(a)$	6
$\tau_{k+1}(\mathbf{a})$	8
$\tau_{k+1}(a, \mathbf{y}, \mathbf{z})$	8
$\mathcal{P}_*(y, z)$	5
$\mathcal{P}_*^k(\mathbf{t})$	6
$H^{(k+1)}(x, \mathbf{y}, \mathbf{z})$	2
$A_{k+1}(N_1, \dots, N_{k+1})$	2
$\mathcal{L}^{(k+1)}(\mathbf{a})$	5
$L^{(k+1)}(\mathbf{a})$	5
$S^{(k+1)}(\mathbf{t})$	15

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## 2. HEURISTIC ARGUMENTS

In this section we develop a heuristic argument in support of Theorem 1.5. The argument given in Subsection 2.1 is a generalization of heuristics developed by Ford in [Fo08a] for the case  $k = 1$  and subsequently by the author in [K10a] for the case  $k \geq 2$ . In Subsection 2.2

we introduce some new ideas in order to explain the appearance of condition (1.1) in the statement of Theorem 1.5.

Before we delve into the details of this argument, we make some definitions and state two elementary but basic results we will be using throughout the entire paper. For  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$  and  $\mathbf{y}, \mathbf{z} \in \mathbb{R}^k$  let

$$\tau_{k+1}(\mathbf{a}) = |\{(d_1, \dots, d_k) \in \mathbb{N}^k : d_1 \cdots d_i | a_1 \cdots a_i \ (1 \leq i \leq k)\}|$$

and

$$\tau_{k+1}(\mathbf{a}, \mathbf{y}, \mathbf{z}) = |\{(d_1, \dots, d_k) \in \mathbb{N}^k : d_1 \cdots d_i | a_1 \cdots a_i, \ y_i < d_i \leq z_i \ (1 \leq i \leq k)\}|.$$

Finally, set

$$\alpha_i = \frac{1}{\log(i+1)} \log \left( \frac{i}{\log(i+1)} \right) \quad (i \in \mathbb{N})$$

and let  $i_0$  be the minimum element of  $\{1, \dots, k\}$  such that

$$|\alpha - \alpha_{k-i_0+1}| = \min\{|\alpha - \alpha_{k-i+1}| : 1 \leq i \leq k\}.$$

**Lemma 2.1.** (a) For  $\mathbf{a} \in \mathbb{N}^k$  we have

$$L^{(k+1)}(\mathbf{a}) \leq \min \left\{ \tau_{k+1}(\mathbf{a})(\log 2)^k, \prod_{i=1}^k (\log a_1 + \cdots + \log a_i + \log 2) \right\}.$$

(b) If  $(a_1 \cdots a_k, b_1 \cdots b_k) = 1$ , then

$$L^{(k+1)}(a_1 b_1, \dots, a_k b_k) \leq \tau_{k+1}(\mathbf{a}) L^{(k+1)}(\mathbf{b}).$$

*Proof.* The proof is similar to the proof of Lemma 3.1 in [Fo08a]. □

**Lemma 2.2.** The sequence  $\{\alpha_i\}_{i \in \mathbb{N}}$  is strictly increasing.

*Proof.* The function

$$\frac{1}{\log(x+1)} \log \left( \frac{x}{\log(x+1)} \right)$$

is easily seen to be strictly increasing for  $x \geq 15$ . Finally, we check numerically that  $\alpha_1 < \alpha_2 < \cdots < \alpha_{15}$ . □

**2.1. Basic set-up and development of the main argument.** Our goal is to understand when an integer  $n \leq x$  is counted by  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$ . We write  $n = a_1 \cdots a_k b$ , where

$$a_i = \prod_{\substack{p^e \parallel n \\ 2y_{i-1} < p \leq 2y_i}} p^e \quad (1 \leq i \leq k).$$

For simplicity, we assume that the numbers  $a_1, \dots, a_k$  are square-free and satisfy  $\log a_i \asymp \log y_i$  for all  $i \in \{1, \dots, k\}$ . Observe that if  $\mathbf{d} = (d_1, \dots, d_k) \in \mathbb{N}^k \cap \prod_{i=1}^k (y_i, 2y_i]$ , then all prime factors of  $d_i$  are at most  $2y_i$  for all  $i \in \{1, \dots, k\}$ . Hence  $\mathbf{d}$  satisfies the relation  $d_1 \cdots d_k | n$  if, and only if,  $d_1 \cdots d_i | a_1 \cdots a_i$  for all  $i \in \{1, \dots, k\}$ . Therefore the integer  $n$  is counted by  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  precisely when  $\tau_{k+1}(\mathbf{a}, \mathbf{y}, 2\mathbf{y}) \geq 1$ . Consider now the set

$$D_{k+1}(\mathbf{a}) = \{(\log d_1, \dots, \log d_k) : d_1 \cdots d_i | a_1 \cdots a_i \ (1 \leq i \leq k)\}.$$



Assume for the moment that  $D_{k+1}(\mathbf{a})$  is well-distributed in  $\prod_{i=1}^k [0, \log(a_1 \cdots a_i)]$ . Then we should have that

$$(2.1) \quad \tau_{k+1}(\mathbf{a}, \mathbf{y}, 2\mathbf{y}) = \left| D_{k+1}(\mathbf{a}) \cap \prod_{i=1}^k (\log y_i, \log y_i + \log 2] \right| \approx \tau_{k+1}(\mathbf{a}) \frac{(\log 2)^k}{\prod_{i=1}^k \log(a_1 \cdots a_i)} \\ \asymp_k \frac{\prod_{i=1}^k (k-i+2)^{\omega(a_i)}}{\prod_{i=1}^k \log y_i}.$$

The right hand side of (2.1) is at least 1 when

$$\sum_{i=1}^k \log(k-i+2)\omega(a_i) \geq \sum_{i=1}^k \log \log y_i + O_k(1) = \sum_{i=1}^k (k-i+1)\ell_i + O_k(1).$$

Since we expect that

$$|\{n \leq x : \omega(a_i) = r_i \ (1 \leq i \leq k)\}| \approx \frac{x}{\log y_k} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!}$$

(see for example [T95, Theorem 4, p. 205]), summing the above relation over all vectors  $\mathbf{r} \in (\mathbb{N} \cup \{0\})^k$  that satisfy

$$(2.2) \quad \sum_{i=1}^k r_i \log(k-i+2) \geq \sum_{i=1}^k \ell_i (k-i+1) + O_k(1)$$

leads to the estimate

$$(2.3) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \approx \frac{x}{\log y_k} \sum_{\substack{\mathbf{r} \in (\mathbb{N} \cup \{0\})^k \\ (2.2)}} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!}.$$

Using Stirling's formula and Lagrange multipliers, we find that the maximum of  $\prod_{i=1}^k \ell_i^{r_i-1} / (r_i-1)!$  under condition (2.2) occurs when  $r_i \sim (k-i+2)^\alpha \ell_i$  (see Section 4 and, in particular, Remark 4.1 and the proof of Lemma 4.2(a)). In fact, Lemma 4.2(a) implies that

$$\sum_{\substack{\mathbf{r} \in (\mathbb{N} \cup \{0\})^k \\ (2.2)}} \frac{\ell_1^{r_1-1} \cdots \ell_k^{r_k-1}}{(r_1-1)! \cdots (r_k-1)!} \asymp_k \frac{\log y_k}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)},$$

so that (2.3) becomes

$$(2.4) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \approx \frac{x}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)}.$$

If  $\beta \gg 1$ , then (2.4) agrees with the conclusion of Theorem 1.5. However, if  $\beta = o_{y_1 \rightarrow \infty}(1)$ , then relation (2.4) overestimates  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  slightly. The problem lies in our assumption that  $D_{k+1}(\mathbf{a})$  is well-distributed. Actually, if  $\beta = o_{y_1 \rightarrow \infty}(1)$ , then with high probability the elements of  $D_{k+1}(\mathbf{a})$  form large clumps. In order to measure the amount of clustering in  $D_{k+1}(\mathbf{a})$ , we use the function  $L^{(k+1)}(\mathbf{a})$ , which we introduced in Section 1. We will show that,

unless the prime factors of  $a_1, \dots, a_k$  satisfy certain constraints, the measure of  $L^{(k+1)}(\mathbf{a})$  is small and, as a consequence, the set  $D_{k+1}(\mathbf{a})$  cannot be well-distributed.

Fix a vector  $\mathbf{r} \in \mathbb{N}^k$  such that

$$(2.5) \quad 0 \leq \sum_{i=1}^k r_i \log(k-i+2) - \sum_{i=1}^k \ell_i(k-i+1) \leq \log(k+1)$$

and  $r_i \sim (k-i+2)^\alpha \ell_i$  as  $\ell_i \rightarrow \infty$ , for all  $i \in \{1, \dots, k\}$ , since most of the contribution to the sum in the right hand side of (2.3) comes from such vectors. Consider  $n$  with  $\omega(a_i) = r_i$  for all  $i \in \{1, \dots, k\}$  and write  $a_i = p_{i,1} \cdots p_{i,r_i}$  with  $2y_{i-1} < p_{i,1} < \cdots < p_{i,r_i} \leq 2y_i$ . Set

$$U_i = 2 \log(k+1) + \sum_{m=1}^{i-1} \ell_m(k-m+1) - \sum_{m=1}^{i-1} r_m \log(k-m+2) \quad (1 \leq i \leq k).$$

Note that

$$\begin{aligned} & \ell_m(k-m+1) - r_m \log(k-m+2) \\ &= (k-m+1 - (k-m+2)^\alpha \log(k-m+2) + o(1)) \ell_m \\ &= \log(k-m+2)((k-m+2)^{\alpha k-m+1} - (k-m+2)^\alpha + o(1)) \ell_m. \end{aligned}$$

So Lemma 2.2 and the definition of  $i_0$  imply that

$$(2.6) \quad U_i \asymp_k \begin{cases} 1 + \ell_1 + \cdots + \ell_{i-1} & \text{if } 1 \leq i \leq i_0, \\ 1 + \ell_i + \cdots + \ell_k & \text{if } i_0 + 1 \leq i \leq k+1, \end{cases}$$

where in the latter case we used (2.5). Assume that there are integers  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, r_i\}$  and a large number  $C$  such that

$$0 \leq \log \log p_{i,j} - \log \log y_{i-1} \leq \frac{\log(k-i+2)j - U_i - C}{k-i+1}.$$

We claim that this causes clustering among the elements of  $D_{k+1}(\mathbf{a})$ . Indeed, if we set  $b_m = a_m$  for  $1 \leq m < i$ ,  $b_i = p_{i,1} \cdots p_{i,j}$  and  $b_m = 1$  for  $i < m \leq k$ , then a double application of Lemma 2.1 implies that

$$\begin{aligned} (2.7) \quad L^{(k+1)}(\mathbf{a}) &\leq \tau_{k+1}(a_1/b_1, \dots, a_k/b_k) L^{(k+1)}(\mathbf{b}) \\ &\leq \left( (k-i+2)^{r_i-j} \prod_{m=i+1}^k (k-m+2)^{r_m} \right) \left( \prod_{m=1}^k \log(2b_1 \cdots b_m) \right) \\ &\ll_k (k-i+2)^{-j} \left( \prod_{m=i}^k (k-m+2)^{r_m} \right) \left( \prod_{m=1}^{i-1} \log y_m \right) (\log y_{i-1} + \log(p_{i,1} \cdots p_{i,j}))^{k-i+1} \\ &\lesssim \left( \prod_{m=i}^k (k-m+2)^{r_m} \right) \left( \prod_{m=1}^{i-1} \log y_m \right) (\log y_{i-1})^{k-i+1} e^{-U_i-C} \asymp_k e^{-C} \prod_{i=1}^k (k-i+2)^{r_i}. \end{aligned}$$

The right hand side of (2.7) is much less than  $\tau_{k+1}(\mathbf{a}) = \prod_{m=1}^k (k-m+2)^{r_m}$  if  $C \rightarrow \infty$ , in which case there must be many elements of  $D_{k+1}(\mathbf{a})$  that are close together. The above

argument suggests that we should focus on numbers  $n$  for which

$$(2.8) \quad \log \log p_{i,j} - \log \log y_{i-1} \geq \frac{\log(k-i+2)j - U_i - O(1)}{k-i+1} \quad (1 \leq i \leq k, R_{i-1} < j \leq R_i).$$

The number of integers  $n$  that satisfy conditions similar to (2.8) was studied by Ford in [Fo07]. Using similar considerations, we find that the probability that an integer  $n$  satisfies (2.8) is about

$$\prod_{i=1}^k \min \left\{ 1, \frac{U_i U_{i+1}}{r_i} \right\} \asymp_k \min \left\{ 1, \frac{(1 + \ell_1 + \cdots + \ell_{i_0-1})(1 + \ell_{i_0+1} + \cdots + \ell_k)}{\ell_{i_0}} \right\},$$

by (2.6). Thus we are led to the refined estimate

$$(2.9) \quad \frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \approx \frac{\min \left\{ 1, \frac{(1 + \ell_1 + \cdots + \ell_{i_0-1})(1 + \ell_{i_0+1} + \cdots + \ell_k)}{\ell_{i_0}} \right\}}{\sqrt{\log \log y_k} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{Q((k-i+2)^\alpha)}}.$$

Finally, we claim that

$$(2.10) \quad \min \left\{ 1, \frac{(1 + \ell_1 + \cdots + \ell_{i_0-1})(1 + \ell_{i_0+1} + \cdots + \ell_k)}{\ell_{i_0}} \right\} \asymp_k \beta.$$

To see this, fix a small parameter  $\delta = \delta(k)$  and observe that if

$$\sum_{i \neq i_1} \ell_i \leq \delta \ell_{i_1} = \delta \max_{1 \leq i \leq k} \ell_i,$$

then

$$(k - i_1 + 2)^\alpha \log(k - i_1 + 2) \ell_{i_1} = (1 + O_k(\delta))(k - i_1 + 1) \ell_{i_1},$$

by the definition of  $\alpha$ . This implies that  $|\alpha - \alpha_{k-i_1+1}| \ll_k \delta$ . So if  $\delta = \delta(k)$  is small enough, then  $i_1 = i_0$  and (2.10) follows immediately. Consider now the case when  $\sum_{i \neq i_1} \ell_i \geq \delta \ell_{i_1}$ . We may also assume that  $i_1 \neq i_0$ ; else, (2.10) holds trivially. Under these assumptions we have that

$$\beta \geq \min \left\{ 1, \frac{\sum_{i \neq i_1} \ell_i}{\ell_{i_1}} \right\} \geq \delta$$

and

$$\min \left\{ 1, \frac{(1 + \ell_1 + \cdots + \ell_{i_0-1})(1 + \ell_{i_0+1} + \cdots + \ell_k)}{\ell_{i_0}} \right\} \geq \min \left\{ 1, \frac{\ell_{i_1}}{\ell_{i_0}} \right\} = 1,$$

which together prove (2.10) in this last case too. By (2.10), we see that (2.9) agrees with the conclusion of Theorem 1.5.

**2.2. Further analysis and optimality of condition (1.1).** Even though the argument given in Subsection 2.1 gives us Theorem 1.5 heuristically, it does not explain the presence of condition (1.1) in the statement of the theorem. This deficiency stems from the fact that the only piece of information we used about  $\mathcal{L}^{(k+1)}(\mathbf{a})$  is Lemma 2.1. In order to understand condition (1.1), we need to pay closer attention to the structure of  $\mathcal{L}^{(k+1)}(\mathbf{a})$ . It turns out that when  $k$  is large, the rich multiplicative structure and the high dimension of the set  $\mathcal{L}^{(k+1)}(\mathbf{a})$  lead to many more bounds on its volume  $L^{(k+1)}(\mathbf{a})$  than just those included in the statement of Lemma 2.1.

**Lemma 2.3.** *Consider integers  $0 = z_0 \leq z_1 \leq \dots \leq z_k \leq z_{k+1} = k$  with  $z_i \geq i - 1$  for all  $i \in \{1, \dots, k\}$ . Let  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$  such that  $\mu^2(a_1 \cdots a_k) = 1$ . Then we have that*

$$L^{(k+1)}(\mathbf{a}) \leq \sum_{\substack{d_j | a_j \\ 1 \leq j \leq k}} \left( \prod_{j=1}^k (z_j - j + 1)^{\omega(d_j)} \right) \\ \times \min \left\{ \prod_{j=0}^k \log^{z_{j+1} - z_j} (2a_1 \cdots a_j), \log^k 2 \prod_{j=1}^k (k - z_j + 1)^{\omega(a_j/d_j)} \right\},$$

with the convention that  $0^0 = 1$ .

*Proof.* Given a  $k$ -tuple  $(d_1, \dots, d_k) \in \mathbb{N}^k$  with  $d_1 \cdots d_i | a_1 \cdots a_i$  for  $1 \leq i \leq k$ , we may uniquely write  $d_i = d_{i,1} d_{i,2} \cdots d_{i,i}$ ,  $1 \leq i \leq k$ , with  $d_{j,j} d_{j+1,j} \cdots d_{k,j} | a_j$  for  $1 \leq j \leq k$ . Thus

$$\mathcal{L}^{(k+1)}(\mathbf{a}) = \bigcup_{\substack{d_{j,j} d_{j+1,j} \cdots d_{k,j} | a_j \\ 1 \leq j \leq k}} \prod_{i=1}^k [\log(d_{i,1} d_{i,2} \cdots d_{i,i}/2), \log(d_{i,1} d_{i,2} \cdots d_{i,i})].$$

For  $i \in \{1, \dots, k\}$  define  $m_i$  as the unique element of  $\{0, 1, \dots, k\}$  such that  $z_{m_i} < i \leq z_{m_i+1}$ . Note that  $i > z_{m_i} \geq m_i - 1$  and thus  $m_i \leq i$ . Set

$$\mathcal{I} = \{(i, j) : 1 \leq j \leq k, j \leq i \leq z_j\} = \{(i, j) : 1 \leq i \leq k, m_i < j \leq i\}.$$

Given numbers  $d_{i,j}$ ,  $(i, j) \in \mathcal{I}$ , with  $d_{j,j} \cdots d_{z_j,j} | a_j$ ,  $1 \leq j \leq k$ , we define the set

$$\begin{aligned} \mathcal{L}(\{d_{i,j} : (i, j) \in \mathcal{I}\}) &= \bigcup_{\substack{d_{i,j}, (i,j) \notin \mathcal{I} \\ 1 \leq j \leq i \leq k \\ d_{z_j+1,j} \cdots d_{k,j} | \frac{a_j}{d_{j,j} \cdots d_{z_j,j}} \quad \forall j}} \prod_{i=1}^k [\log(d_{i,1} \cdots d_{i,i}/2), \log(d_{i,1} \cdots d_{i,i})] \\ &= \bigcup_{\substack{d_{z_j+1,j} \cdots d_{k,j} | \frac{a_j}{d_{j,j} \cdots d_{z_j,j}} \\ 1 \leq j \leq k}} \prod_{i=1}^k [\log(d_{i,1} \cdots d_{i,m_i}/2), \log(d_{i,1} \cdots d_{i,m_i})] \\ &\quad + (\log(d_{1,m_1+1} \cdots d_{1,1}), \log(d_{2,m_2+1} \cdots d_{2,2}), \dots, \log(d_{k,m_k+1} \cdots d_{k,k})). \end{aligned}$$

The above identity implies that

$$\text{Vol}(\mathcal{L}(\{d_{i,j} : (i,j) \in \mathcal{I}\})) \leq \min \left\{ \prod_{i=1}^k \log(2a_1 \cdots a_{m_i}), (\log 2)^k \prod_{i=1}^k (k - z_i + 1)^{\omega\left(\frac{a_i}{d_{i,i} \cdots d_{z_i,i}}\right)} \right\}.$$

Since

$$\mathcal{L}^{(k+1)}(\mathbf{a}) = \bigcup_{\substack{d_{i,j}, (i,j) \in \mathcal{I} \\ d_{j,j} \cdots d_{z_j,j} | a_j \quad \forall j}} \mathcal{L}(\{d_{i,j} : (i,j) \in \mathcal{I}\}),$$

we find that

$$\begin{aligned} L^{(k+1)}(\mathbf{a}) &\leq \sum_{\substack{d_{i,j}, (i,j) \in \mathcal{I} \\ D_j = d_{j,j} \cdots d_{z_j,j} | a_j \quad \forall j}} \min \left\{ \prod_{i=1}^k \log(2a_1 \cdots a_{m_i}), (\log 2)^k \prod_{i=1}^k (k - z_i + 1)^{\omega(a_i/D_i)} \right\} \\ &= \sum_{\substack{D_j | a_j \\ 1 \leq j \leq k}} \left( \prod_{i=1}^k (z_i - i + 1)^{\omega(D_i)} \right) \\ &\quad \times \min \left\{ \prod_{i=1}^k \log(2a_1 \cdots a_{m_i}), (\log 2)^k \prod_{i=1}^k (k - z_i + 1)^{\omega(a_i/D_i)} \right\}. \end{aligned}$$

To complete the proof of the lemma note that

$$\prod_{i=1}^k \log(2a_1 \cdots a_{m_i}) = \prod_{j=0}^k \log^{z_{j+1} - z_j} (2a_1 \cdots a_j).$$

□

Using the above lemma, we show that condition (1.1) is optimal, that is to say, for every fixed  $\gamma$  such that

$$(2.11) \quad \frac{1}{\log 2} \log \frac{1}{\log 2} < \gamma < 1 - \frac{1}{\log(k+1)} \log \left( \frac{(k+1) \log(k+1) - 2 \log 2}{k-1} \right),$$

there are choices of  $y_1, \dots, y_k$  such that  $\alpha = \alpha(k; \mathbf{y}) = \gamma$  and

$$(2.12) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) = o \left( \frac{\beta}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)} \right) \quad (y_1 \rightarrow \infty).$$

The argument we give is heuristic but, if combined with the results of Sections 4 and 5, it can be made rigorous.

The right inequality in (2.11) is equivalent to

$$(2.13) \quad \frac{(k+1) \log(k+1) - 2 \log 2}{(k+1)^{1-\gamma}} < k-1.$$

Also, inequalities (1.3) and (2.11) imply that

$$k - (k+1)^\gamma \log(k+1) > 0 \quad \text{and} \quad 2^\gamma \log 2 - 1 > 0.$$

So if we set select  $y_1 = y_2 = \dots = y_{k-1}$  large enough, then there is a unique  $y_k \geq y_{k-1}$  such that

$$\ell_k = \frac{1}{2^\gamma \log 2 - 1} \sum_{i=1}^{k-1} (k - i + 1 - (k - i + 2)^\gamma \log(k - i + 2)) \ell_i,$$

that is to say, there is a unique  $y_k$  so that the  $k$ -tuple  $\mathbf{y} = (y_1, \dots, y_k)$  satisfies the relation  $\alpha(k, \mathbf{y}) = \gamma$ . We claim that (2.12) holds and we support this claim with a heuristic argument:

Similarly to Subsection 2.1, we consider  $\mathbf{a} = (a_1, \dots, a_k)$  such that  $a_i \in \mathcal{P}_*(2y_{i-1}, 2y_i)$ . Note that we necessarily have that  $a_2 = \dots = a_{k-1} = 1$ . Set  $r_i = \omega(a_i)$  for all  $i \in \{1, \dots, k\}$  and assume further that  $\log a_i \asymp \log y_i$  for  $i \in \{1, k\}$ , that

$$(2.14) \quad r_i \sim (k - i + 2)^\alpha \ell_i = (k - i + 2)^\gamma \ell_i \quad (i \in \{1, k\}, y_1 \rightarrow \infty)$$

and that (2.5) holds. We will show that

$$(2.15) \quad L^{(k+1)}(\mathbf{a}) = o\left(\prod_{i=1}^k \log y_i\right) \quad (y_1 \rightarrow \infty).$$

Indeed, Lemma 2.3, applied with  $z_1 = \dots = z_k = k - 1$ , implies that

$$\begin{aligned} L^{(k+1)}(\mathbf{a}) &\ll_k \sum_{\substack{d_k | a_k \\ 1 \leq j \leq k}} \left( \prod_{j=1}^k (k - j)^{\omega(d_j)} \right) \min \left\{ \log y_k, \prod_{j=1}^k 2^{\omega(a_j/d_j)} \right\} \\ &= \sum_{d_1 | a_1} (k - 1)^{\omega(d_1)} \min \left\{ \log y_k, 2^{r_k + \omega(a_1/d_1)} \right\} \end{aligned}$$

(note that all summands with  $d_k > 1$  vanish and  $d_i = a_i = 1$  for  $i \in \{2, \dots, k - 1\}$ ). The main contribution to the sum

$$\sum_{d_1 | a_1} (k - 1)^{\omega(d_1)} 2^{r_k + \omega(a_1/d_1)} = (k + 1)^{r_1} 2^{r_k} = \prod_{j=1}^k (k - j + 2)^{r_j} \asymp_k \prod_{i=1}^k \log y_i$$

comes from integers  $d_1$  such that

$$(2.16) \quad \omega(d_1) \sim \frac{k - 1}{k + 1} \cdot r_1 \quad (y_1 \rightarrow \infty).$$

If  $d_1$  satisfies (2.16), then relations (2.5), (2.13) and (2.14) and the fact that  $r_i = 0$  and  $\ell_i = O(1)$  for  $i \in \{2, \dots, k - 1\}$  imply that

$$\begin{aligned} (r_k + \omega(a_1/d_1)) \log 2 - \log \log y_k &= \frac{2 \log 2}{k + 1} r_1 + (\log 2) r_k - \ell_1 - \ell_k + o_k(\ell_1) \\ &= (k - 1) \ell_1 - \frac{(k + 1) \log(k + 1) - 2 \log 2}{k + 1} r_1 + o_k(\ell_1) \rightarrow +\infty \end{aligned}$$

as  $y_1 \rightarrow \infty$ . Consequently, for integers  $d_1$  that satisfy (2.16) we have that

$$\min \left\{ \log y_k, 2^{r_k + \omega(a_1/d_1)} \right\} = \log y_k = o\left(2^{r_k + \omega(a_1/d_1)}\right) \quad (y_1 \rightarrow \infty),$$

which in turn implies that relation (2.15) is indeed true. This yields that, in contrast to the prediction of the arguments in Subsection 2.1,  $D_{k+1}(\mathbf{a})$  is not well-distributed for such  $\mathbf{a}$ . Hence, in general, relation (2.9) overestimates the size of  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$ .

*Remark 2.1.* The information about  $L^{(k+1)}(\mathbf{a})$  that is contained in Lemma 2.3 makes its appearance implicitly in the statement of Lemma 6.2. An approach that could potentially extend Theorem 1.5 to the case when condition (1.1) fails is to insert Lemma 2.3 into the proof of the upper bound in Theorem 1.5 (Section 5) and then adjust the lower bound argument accordingly (Sections 6, 7 and 8).

### 3. LOCAL-TO-GLOBAL ESTIMATES

In this section we reduce the counting in  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  to the estimation of

$$S^{(k+1)}(\mathbf{t}) := \sum_{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{t})} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k}$$

and prove Theorem 1.7. This reduction has also been carried in the author's thesis [K10b], but we give it here for completeness. The basic ideas behind it can be found in [Fo08a] and [K10a]. However, the details are more complicated, especially in the proof of the upper bound implicit in Theorem 1.7, because of the presence of more parameters. Finally, we employ Theorem 1.7 to deduce Theorem 1.1 in Subsection 3.4.

*Remark 3.1.* In order to show Theorem 1.7 for some  $k \geq 1$ , we may assume without loss of generality that  $y_1 > C'_k$ , where  $C'_1, C'_2, \dots, C'_k, \dots$  is an increasing sequence of large constants. Indeed, suppose for the moment that Theorem 1.7 holds for all  $k \geq 1$  when  $y_1 > C'_k$  and consider the case when  $y_1 \leq C'_k$ . Then either  $y_k \leq C'_k$ , in which case Theorem 1.7 follows immediately, or there exists  $l \in \{1, \dots, k-1\}$  such that  $y_l \leq C'_k < y_{l+1}$ . In the latter case let  $\mathbf{y}' = (y_{l+1}, \dots, y_k)$  and  $d = \lfloor 2y_1 \rfloor \cdots \lfloor 2y_l \rfloor \leq 2^l y_1 \cdots y_l \leq (2C'_k)^k$  and note that

$$H^{(k-l+1)}\left(\frac{x}{d}, \mathbf{y}', 2\mathbf{y}'\right) \leq H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \leq H^{(k-l+1)}(x, \mathbf{y}', 2\mathbf{y}'),$$

Moreover,

$$\frac{x/d}{y_{l+1} \cdots y_k} \geq \frac{x}{2^l y_1 \cdots y_k} \geq 2^{k-l} y_k.$$

So the desired bound on  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  follows by Theorem 1.7 applied to  $H^{(k-l+1)}(x, \mathbf{y}', 2\mathbf{y}')$  and  $H^{(k-l+1)}(x/d, \mathbf{y}', 2\mathbf{y}')$ , which holds since  $y_{l+1} > C'_k \geq C'_{k-l}$ .

**3.1. Auxiliary results.** Before we launch into the proof of Theorem 1.7, we list a few results from number theory and analysis that we shall need. First, we state a standard sieve estimate for easy reference (see for example [HT, Theorem 06]).

**Lemma 3.1.** *For  $4 \leq 2z \leq x$  we have*

$$|\{n \leq x : P^-(n) > z\}| \asymp \frac{x}{\log z}.$$

Next, we have the following result, which follows by Lemma 2.3(b) in [K10a].

**Lemma 3.2.** *Let  $f : \mathbb{N} \rightarrow [0, +\infty)$  be an arithmetic function satisfying  $f(ap) \leq C_f f(a)$  for all integers  $a$  and all primes  $p$  with  $(a, p) = 1$ , where  $C_f$  is a positive constant depending only on  $f$ . Let  $h \geq 0$  and  $3/2 \leq y \leq x \leq z^C$  for some  $C > 0$ . Then*

$$\sum_{\substack{a \in \mathcal{P}_*(y, x) \\ a > z}} \frac{f(a)}{a \log^h(P^+(a))} \ll_{C_f, h, C} \exp \left\{ -\frac{\log z}{2 \log x} \right\} \frac{1}{\log^h x} \sum_{a \in \mathcal{P}_*(y, x)} \frac{f(a)}{a}.$$

Finally, we need a covering lemma which is a slightly different version of Lemma 3.15 in [F]. If  $r$  is a positive real number and  $I$  is a  $k$ -dimensional rectangle, then  $rI$  will denote the rectangle which has the same center with  $I$  and  $r$  times its diameter. More formally, if  $\mathbf{x}_0$  is the center of  $I$ , then  $rI := \{r(\mathbf{x} - \mathbf{x}_0) + \mathbf{x}_0 : \mathbf{x} \in I\}$ . The lemma is then formulated as follows:

**Lemma 3.3.** *Let  $I_1, \dots, I_N$  be  $k$ -dimensional cubes of the form  $[a_1, b_1) \times \dots \times [a_k, b_k)$  ( $b_1 - a_1 = \dots = b_k - a_k > 0$ ). Then there exists a sub-collection  $I_{i_1}, \dots, I_{i_M}$  of mutually disjoint cubes such that*

$$\bigcup_{n=1}^N I_n \subset \bigcup_{m=1}^M 3I_{i_m}.$$

**3.2. The lower bound in Theorem 1.7.** We start with the proof of the lower bound implicit in Theorem 1.7, which is simpler. First, we prove a weaker result; then we use Lemma 3.2 to complete the proof. Note that the lemma below is similar to Lemma 2.1 in [Fo08a], Lemma 4.1 in [Fo08b] and Lemma 3.2 in [K10a].

**Lemma 3.4.** *Let  $k \geq 1$ ,  $x \geq 1$  and  $3 = y_0 \leq y_1 \leq y_2 \leq \dots \leq y_k$  with  $2^k y_1 \dots y_k \leq x/y_k$  and  $y_1 > C'_k$ . Then*

$$(3.1) \quad \frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \gg_k \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-(k-i+2)} \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y}) \\ a_i \leq y_i^{1/8k} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k}.$$

*Proof.* Set

$$x' = \frac{x}{2^k y_1 \dots y_k} \geq y_k.$$

Consider integers  $n = a_1 \dots a_k p_1 \dots p_k b \leq x$  such that

- (1)  $\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y})$  and  $a_i \leq y_i^{1/(8k)}$  for  $i = 1, \dots, k$ ;
- (2)  $p_1, \dots, p_k$  are prime numbers with  $(\log(y_1/p_1), \dots, \log(y_k/p_k)) \in \mathcal{L}^{(k+1)}(\mathbf{a})$ ;
- (3) If  $x' \leq y_k^2$ , then let  $b$  be a prime number  $> y_k^{1/8}$ ; if  $x' > y_k^2$ , then let  $b$  be an integer with  $P^-(b) > 2y_k$ .

Note that for every  $i \in \{1, \dots, k\}$  all prime factors of  $a_i$  lie in  $(2y_{i-1}, y_i^{1/(8k)})$ . Also, condition (2) in the definition of  $n$  is equivalent to the existence of integers  $d_1, \dots, d_k$  such that  $d_1 \dots d_i | a_1 \dots a_i$  and  $y_i/p_i < d_i \leq 2y_i/p_i$  for all  $i \in \{1, \dots, k\}$ . In particular,  $\tau_{k+1}(n, \mathbf{y}, 2\mathbf{y}) \geq 1$ . Furthermore, we have that

$$y_i^{7/8} \leq \frac{y_i}{a_1 \dots a_i} \leq \frac{y_i}{d_i} < p_i \leq 2 \frac{y_i}{d_i} \leq 2y_i.$$



So  $(a_1 \cdots a_k, p_1 \cdots p_k b) = 1$  and hence this representation of  $n$ , if it exists, it is unique up to a possible permutation of  $p_1, \dots, p_k$  and the prime factors of  $b$  that lie in  $(y_1^{7/8}, 2y_k]$ . Since  $b$  has at most one such prime factor,  $n$  has a bounded number of such representations. Fix  $a_1, \dots, a_k$  and  $p_1, \dots, p_k$  and note that

$$(3.2) \quad X := \frac{x}{a_1 \cdots a_k p_1 \cdots p_k} \geq \frac{x'}{y_k^{1/8}} \geq (x')^{7/8} > 2y_k^{1/8}.$$

We start by counting the number of possibilities for  $b$ . We consider two cases. First, if  $x' > y_k^2$ , then  $X > 4y_k$ , by (3.2), provided that  $C'_k$  is large enough. So Lemma 3.1 implies that

$$\sum_{b \text{ admissible}} 1 = \sum_{b \leq X, P^-(b) > 2y_k} 1 \gg_k \frac{X}{\log y_k},$$

by Lemma 3.1. On the other hand, if  $x' \leq y_k^2$ , then

$$X = \frac{x}{a_1 \cdots a_k p_1 \cdots p_k} \leq \frac{x d_1 \cdots d_k}{a_1 \cdots a_k y_1 \cdots y_k} \leq 2^k x' \leq 2^k y_k^2.$$

The above inequality and (3.2) imply that

$$\sum_{b \text{ admissible}} 1 = \sum_{\substack{y_k^{1/8} < b \leq X \\ b \text{ prime}}} 1 \geq \sum_{\substack{X/2 < b \leq X \\ b \text{ prime}}} 1 \gg \frac{X}{\log X} \gg_k \frac{X}{\log y_k}.$$

In any case, we have that

$$\sum_{b \text{ admissible}} 1 \gg_k \frac{X}{\log y_k}$$

and, consequently,

$$(3.3) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \gg_k \frac{x}{\log y_k} \sum_{\substack{\mathbf{a} \in \mathcal{P}_*(2\mathbf{y}) \\ a_i \leq y_i^{1/8k} (1 \leq i \leq k)}} \frac{1}{a_1 \cdots a_k} \sum_{(\log \frac{y_1}{p_1}, \dots, \log \frac{y_k}{p_k}) \in \mathcal{L}^{(k+1)}(\mathbf{a})} \frac{1}{p_1 \cdots p_k}.$$

Fix  $\mathbf{a} \in \mathcal{P}_*(2\mathbf{y})$  with  $a_i \leq y_i^{1/(8k)}$  for  $i = 1, \dots, k$ . Let  $\{I_r\}_{r=1}^R$  be the collection of cubes  $[\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$  with  $d_1 \cdots d_i | a_1 \cdots a_i$ ,  $1 \leq i \leq k$ . Then for  $I = [\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$  in this collection we have that

$$\sum_{(\log \frac{y_1}{p_1}, \dots, \log \frac{y_k}{p_k}) \in I} \frac{1}{p_1 \cdots p_k} = \prod_{i=1}^k \sum_{y_i/d_i < p_i \leq 2y_i/d_i} \frac{1}{p_i} \gg_k \frac{1}{\log y_1 \cdots \log y_k},$$

because  $d_i \leq a_1 \cdots a_i \leq y_i^{1/8}$  for  $1 \leq i \leq k$ . By Lemma 3.3 there exists a sub-collection  $\{I_{r_s}\}_{s=1}^S$  of mutually disjoint cubes so that

$$S(3 \log 2)^k \geq \text{Vol} \left( \bigcup_{s=1}^S 3I_{r_s} \right) \geq \text{Vol} \left( \bigcup_{r=1}^R I_r \right) = L^{(k+1)}(\mathbf{a}).$$

Hence

$$\begin{aligned} \sum_{(\log \frac{y_1}{p_1}, \dots, \log \frac{y_k}{p_k}) \in \mathcal{L}^{(k+1)}(\mathbf{a})} \frac{1}{p_1 \cdots p_k} &\geq \sum_{s=1}^S \sum_{(\log \frac{y_1}{p_1}, \dots, \log \frac{y_k}{p_k}) \in I_{r_s}} \frac{1}{p_1 \cdots p_k} \gg_k \frac{S}{\log y_1 \cdots \log y_k} \\ &\gg_k \frac{L^{(k+1)}(\mathbf{a})}{\log y_1 \cdots \log y_k}. \end{aligned}$$

Combining the above estimate with (3.3) completes the proof of the lemma.  $\square$

Having proven the above lemma, it is not so hard to finish the proof of the lower bound of Theorem 1.7. We give the argument below.

*Proof of Theorem 1.7 (lower bound).* For every fixed  $i \in \{1, \dots, k\}$  and integers  $a_1, \dots, a_{i-1}$  and  $a_{i+1}, \dots, a_k$ , the function  $a_i \rightarrow L^{(k+1)}(\mathbf{a})$  satisfies the hypothesis of Lemma 3.2 with  $C_f = k - i + 2 \leq k + 1$ , by Lemma 2.1(b). So if we set

$$\mathcal{P} = \left\{ \mathbf{a} \in \mathbb{N}^k : a_i \in \mathcal{P} \left( 2y_{i-1}, y_i^{1/M} \right) \ (1 \leq i \leq k) \right\}$$

for some sufficiently large  $M = M(k)$ , then

$$\begin{aligned} \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y}) \\ a_i \leq y_i^{1/(8k)} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} &\geq \sum_{\substack{\mathbf{a} \in \mathcal{P} \\ a_i \leq y_i^{1/(8k)} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} = \sum_{\mathbf{a} \in \mathcal{P}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \left( 1 + O_k \left( e^{-\frac{M}{16k}} \right) \right) \\ &\geq \frac{1}{2} \sum_{\mathbf{a} \in \mathcal{P}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k}. \end{aligned}$$

By the above inequality and Lemma 2.1(b), we deduce that

$$S^{(k+1)}(\mathbf{y}) \leq \sum_{\mathbf{a} \in \mathcal{P}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \prod_{i=1}^k \sum_{\substack{b_i \in \mathcal{P}_*(y_{i-1}, 2y_{i-1}) \\ \text{or } b_i \in \mathcal{P}_*(y_i^{1/M}, y_i)}} \frac{\tau_{k-i+2}(b_i)}{b_i} \ll_k \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y}) \\ a_i \leq y_i^{1/(8k)} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k}.$$

Combining the above estimate with Lemma 3.4 completes the proof of the lower bound in Theorem 1.7.  $\square$

**3.3. The upper bound in Theorem 1.7.** In this subsection we complete the proof of Theorem 1.7. Before we proceed to the proof, we need to define some auxiliary notation. For  $\mathbf{y}, \mathbf{z} \in \mathbb{R}^k$  and  $x \geq 1$  set

$$H_*^{(k+1)}(x, \mathbf{y}, \mathbf{z}) = |\{n \leq x : \mu^2(n) = 1, \exists d_1 \cdots d_k | n \text{ such that } y_i < d_i \leq z_i \ (1 \leq i \leq k)\}|.$$

Also, for  $\mathbf{t} \in [1, +\infty)^k$ ,  $\mathbf{h} \in [0, +\infty)^k$  and  $\epsilon > 0$  define

$$\mathcal{P}_*^k(\mathbf{t}; \epsilon) = \left\{ \mathbf{a} \in \mathbb{N}^k : a_i \in \mathcal{P}_* \left( \max \left\{ P^+(a_1 \cdots a_{i-1}), \frac{t_{i-1}^\epsilon}{a_1 \cdots a_{i-1}} \right\}, t_i \right) \ (1 \leq i \leq k) \right\},$$

where  $t_0 = 1$ , and

$$S^{(k+1)}(\mathbf{t}; \mathbf{h}, \epsilon) = \sum_{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{t}; \epsilon)} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \prod_{i=1}^k \log^{-h_i} \left( P^+(a_1 \cdots a_i) + \frac{t_i^\epsilon}{a_1 \cdots a_i} \right).$$

Lastly, let

$$\mathbf{e}_k = (e_{k,1}, \dots, e_{k,k}) = (\underbrace{1, \dots, 1}_{k-1 \text{ times}}, 2) \in \mathbb{R}^k.$$

Then we have the following estimate.

**Lemma 3.5.** *Let  $\sqrt{C'_k} \leq y_1 \leq \cdots \leq y_k \leq x$  with  $2^{k+1}y_1 \cdots y_k \leq x/(2y_k)^{7/8}$ . Then*

$$H_*^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) - H_*^{(k+1)}(x/2, \mathbf{y}, 2\mathbf{y}) \ll_k x S^{(k+1)}(2\mathbf{y}; \mathbf{e}_k, 7/8).$$

*Proof.* Let  $n \in (x/2, x]$  be a square-free integer for which there exist integers  $d_i \in (y_i, 2y_i]$ ,  $1 \leq i \leq k$ , with  $d_1 \cdots d_k | n$ . If we set  $d_{k+1} = n/(d_1 \cdots d_k)$  and  $y_{k+1} = x/(2^{k+1}y_1 \cdots y_k)$ , then we have that  $n = d_1 \cdots d_{k+1}$  with  $y_i < d_i \leq 2^{k+1}y_i$  for  $1 \leq i \leq k+1$ . Let  $z_1, \dots, z_{k+1}$  be the sequence  $y_1, \dots, y_{k+1}$  ordered increasingly. Also, let  $\sigma$  be the unique permutation in  $S_{k+1}$  for which  $P^+(d_{\sigma(1)}) < \cdots < P^+(d_{\sigma(k+1)})$  and set  $p_j = P^+(d_{\sigma(j)})$  for  $1 \leq j \leq k+1$  and  $p_0 = 1$ . We can write  $n = a_1 \cdots a_k p_1 \cdots p_k b$  with  $P^-(b) > p_k$  and  $a_i \in \mathcal{P}_*(p_{i-1}, p_i)$  for all  $1 \leq i \leq k$ . We claim that

$$(3.4) \quad p_i > Q_i := \max \left\{ P^+(a_1 \cdots a_i), \frac{(2y_i)^{7/8}}{a_1 \cdots a_i} \right\} \quad (1 \leq i \leq k).$$

Indeed, for every  $j \in \{1, \dots, k\}$  we have that  $y_{\sigma(j)} < d_{\sigma(j)} = p_j d$  for some  $d | a_1 \cdots a_j$  and therefore  $y_{\sigma(j)} < p_j a_1 \cdots a_j$ . Consequently,

$$p_i = \max_{1 \leq j \leq i} p_j > \max_{1 \leq j \leq i} \frac{y_{\sigma(j)}}{a_1 \cdots a_j} \geq \frac{\max_{1 \leq j \leq i} y_{\sigma(j)}}{a_1 \cdots a_i} \geq \frac{z_i}{a_1 \cdots a_i} \geq \frac{(2y_i)^{7/8}}{a_1 \cdots a_i} \quad (1 \leq i \leq k),$$

by the definition of  $z_1, \dots, z_{k+1}$  and our assumption that  $y_1 \leq \cdots \leq y_k \leq \frac{1}{2}y_{k+1}^{8/7}$ . Moreover,

$$p_i = \max_{1 \leq j \leq i} p_j > \max_{1 \leq j \leq i} P^+(a_j) = P^+(a_1 \cdots a_j).$$

So (3.4) follows. In addition,

$$P^+(a_i) < p_i = P^+(d_{\sigma(i)}) \leq \max_{1 \leq j \leq i} P^+(d_j) \leq 2y_i \quad (1 \leq i \leq k),$$

by the choice of  $\sigma$ , and

$$P^-(a_i) > p_{i-1} > Q_{i-1} \quad (2 \leq i \leq k),$$

by (3.4). In particular,  $\mathbf{a} = (a_1, \dots, a_k) \in \mathcal{P}_*^k(2\mathbf{y}; 7/8)$ . Furthermore, note that

$$(d_{\sigma(1)}/p_1) \cdots (d_{\sigma(i)}/p_i) | a_1 \cdots a_i \quad \text{and} \quad \frac{y_{\sigma(i)}}{p_i} < \frac{d_{\sigma(i)}}{p_i} \leq \frac{2^{k+1}y_{\sigma(i)}}{p_i} \quad (1 \leq i \leq k),$$

that is to say, there are numbers  $w_1, \dots, w_k \in \{1, 2, 2^2, \dots, 2^k\}$  such that

$$(3.5) \quad \left( \log \frac{w_1 y_{\sigma(1)}}{p_1}, \dots, \log \frac{w_k y_{\sigma(k)}}{p_k} \right) \in \mathcal{L}^{(k+1)}(\mathbf{a}).$$

Lastly, observe that  $p_{k+1}|b$  and consequently  $b \geq p_{k+1} > p_k > Q_k$ , by (3.4). Similarly, we have  $P^-(b) > p_k > Q_k$ . Combining all of the above, we deduce that

$$\begin{aligned}
 & H_*^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) - H_*^{(k+1)}(x/2, \mathbf{y}, 2\mathbf{y}) \\
 & \leq \sum_{\sigma \in S_{k+1}} \sum_{\substack{w_i \in \{1, 2, \dots, 2^k\} \\ 1 \leq i \leq k}} \sum_{\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y}; 7/8)} \sum_{\substack{p_1, \dots, p_k \\ (3.4), (3.5)}} \sum_{\substack{Q_k < b \leq x/(p_1 \cdots p_k) \\ P^-(b) > Q_k}} 1 \\
 (3.6) \quad & \ll_k \sum_{\sigma \in S_{k+1}} \sum_{\substack{w_i \in \{1, 2, \dots, 2^k\} \\ 1 \leq i \leq k}} \sum_{\mathbf{a} \in \mathcal{P}_*^k(2\mathbf{y}; 7/8)} \sum_{\substack{p_1, \dots, p_k \\ (3.4), (3.5)}} \frac{x}{a_1 \cdots a_k p_1 \cdots p_k \log Q_k},
 \end{aligned}$$

by Lemma 3.1. We fix  $\sigma, w_1, \dots, w_k$  and  $a_1, \dots, a_k$  as above and estimate the sum over the primes  $p_1, \dots, p_k$  in the right hand side of (3.6). In order to analyze condition (3.5), consider the collection  $\{I_r\}_{r=1}^R$  of cubes of the form  $[\log(m_1/2), \log m_1) \times \cdots \times [\log(m_k/2), \log m_k)$  with  $m_1 \cdots m_i | a_1 \cdots a_i$  for  $1 \leq i \leq k$ . By Lemma 3.3, there is a sub-collection  $\{I_{r_s}\}_{s=1}^S$  of mutually disjoint such cubes for which  $\mathcal{L}^{(k+1)}(\mathbf{a}) \subset \bigcup_{s=1}^S 3I_{r_s}$ . Consider  $I_{r_s} = [\log(m_1/2), \log m_1) \times \cdots \times [\log(m_k/2), \log m_k)$  in this sub-collection and set

$$U_i = \frac{w_i y_{\sigma(i)}}{2m_i} \quad (1 \leq i \leq k).$$

Then we find that

$$(3.7) \quad \left( \log \frac{w_1 y_{\sigma(1)}}{p_1}, \dots, \log \frac{w_k y_{\sigma(k)}}{p_k} \right) \in 3I_{r_s} = \left[ \log \frac{m_1}{4}, \log(2m_1) \right) \times \cdots \times \left[ \log \frac{m_k}{4}, \log(2m_k) \right)$$

if, and only if,  $U_i < p_i \leq 8U_i$  for all  $i = 1, \dots, k$ . So

$$\sum_{\substack{p_1, \dots, p_k \\ (3.4), (3.7)}} \frac{1}{p_1 \cdots p_k} \leq \prod_{i=1}^k \sum_{\substack{U_i < p_i \leq 8U_i \\ p_i > Q_i}} \frac{1}{p_i} \ll_k \prod_{i=1}^k \frac{1}{\log(\max\{U_i, Q_i\})} \leq \prod_{i=1}^k \frac{1}{\log Q_i}.$$

Therefore we deduce that

$$\sum_{\substack{p_1, \dots, p_k \\ (3.4), (3.5)}} \frac{1}{p_1 \cdots p_k} \leq \sum_{s=1}^S \sum_{\substack{p_1, \dots, p_k \\ (3.4), (3.7)}} \frac{1}{p_1 \cdots p_k} \ll_k \frac{S}{\log Q_1 \cdots \log Q_k} \leq \frac{L^{(k+1)}(\mathbf{a})}{(\log 2)^k \log Q_1 \cdots \log Q_k}.$$

Inserting the above estimate into (3.6) completes the proof of the lemma.  $\square$

Next, we bound the sum  $S^{(k+1)}(\mathbf{t}; \mathbf{h}, \epsilon)$  from above in terms of  $S^{(k+1)}(\mathbf{t})$ . This is accomplished by establishing an iterative inequality that simplifies the complicated range of summation  $\mathcal{P}_*^k(\mathbf{t}; \epsilon)$  by gradually reducing it to the much simpler set  $\mathcal{P}_*^k(\mathbf{t})$  and, at the same time, eliminates the complicated logarithms that appear in the summands of  $S^{(k+1)}(\mathbf{t}; \mathbf{h}, \epsilon)$ . Lemma 3.2 plays a crucial role in the proof of this inequality

**Lemma 3.6.** Fix  $k \geq 1$ ,  $\epsilon > 0$  and  $\mathbf{h} = (h_1, \dots, h_k) \in [0, +\infty)^k$ . For  $\mathbf{t} = (t_1, \dots, t_k)$  with  $3 \leq t_1 \leq \dots \leq t_k$  we have that

$$S^{(k+1)}(\mathbf{t}; \mathbf{h}, \epsilon) \ll_{k, \mathbf{h}, \epsilon} \left( \prod_{i=1}^k \log^{-h_i} t_i \right) S^{(k+1)}(\mathbf{t}).$$

*Proof.* Set  $\delta = \epsilon/(2k)$  and  $t_0 = 1$ . For  $l \in \{1, \dots, k\}$  define

$$h_{l,i} = \begin{cases} h_i & \text{if } i \in \{1, \dots, l-1\} \cup \{k\}, \\ h_i + k - i + 1 & \text{if } l \leq i \leq k-1, \end{cases}$$

and

$$\mathcal{P}_l(\mathbf{t}) = \left\{ \mathbf{a} \in \mathbb{N}^k : a_i \in \mathcal{P}_* \left( \max \left\{ P^+(a_1 \cdots a_{i-1}), t_{i-1}^{\epsilon/2+l\delta} / (a_1 \cdots a_{i-1}) \right\}, t_i \right) \ (1 \leq i \leq l), \right. \\ \left. a_i \in \mathcal{P}_*(t_{i-1}, t_i) \ (l+1 \leq i \leq k) \right\}.$$

Also, let  $h_{0,i} = h_{1,i}$  for  $i \in \{1, \dots, k\}$  and  $\mathcal{P}_0(\mathbf{t}) = \mathcal{P}_1(\mathbf{t})$ . Lastly, for  $l \in \{0, \dots, k\}$  set  $\mathbf{h}_l = (h_{l,1}, \dots, h_{l,k})$  and

$$\tilde{S}_l^{(k+1)}(\mathbf{t}; \mathbf{h}_l) = \sum_{\mathbf{a} \in \mathcal{P}_l(\mathbf{t})} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \prod_{i=1}^l \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_i) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_i} \right) \\ \times \prod_{i=l+1}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \right).$$

Note that

$$(3.8) \quad \tilde{S}_k^{(k+1)}(\mathbf{t}; \mathbf{h}_k) = S^{(k+1)}(\mathbf{t}; \mathbf{h}, \epsilon)$$

and

$$(3.9) \quad \tilde{S}_0^{(k+1)}(\mathbf{t}; \mathbf{h}_0) \asymp_{k, \epsilon, \mathbf{h}} \left( \prod_{i=1}^k (\log t_i)^{-h_{0,i}} \right) S^{(k+1)}(\mathbf{t}) \\ = \left( \prod_{i=1}^{k-1} (\log t_i)^{-(h_i + k - i + 1)} \right) (\log t_k)^{-h_k} S^{(k+1)}(\mathbf{t}).$$

We claim that

$$(3.10) \quad \tilde{S}_l^{(k+1)}(\mathbf{t}; \mathbf{h}_l) \ll_{k, \mathbf{h}, \epsilon} (\log 2t_{l-1})^{k-l+2} \tilde{S}_{l-1}^{(k+1)}(\mathbf{t}; \mathbf{h}_{l-1}) \quad (1 \leq l \leq k).$$

Clearly, if we prove (3.10), then the lemma follows immediately by iterating (3.10) and combining the resulting inequality with relations (3.8) and (3.9). So we fix  $l \in \{1, \dots, k\}$  and proceed to the proof of (3.10). Consider integers  $a_1, \dots, a_{l-1}$  such that

$$a_i \in \mathcal{P}_* \left( \max \left\{ P^+(a_1 \cdots a_{i-1}), \frac{t_{i-1}^{\epsilon/2+l\delta}}{a_1 \cdots a_{i-1}} \right\}, t_i \right) \quad (1 \leq i \leq l-1)$$

and  $a_{l+1}, \dots, a_k$  such that

$$a_i \in \mathcal{P}_*(t_{i-1}, t_i) \quad (l+1 \leq i \leq k)$$

and set

$$t'_{l-1} = \max \left\{ P^+(a_1 \cdots a_{l-1}), \frac{t_{l-1}^{\epsilon/2+l\delta}}{a_1 \cdots a_{l-1}} \right\}.$$

Observe that in order to show (3.10) it suffices to prove that

$$(3.11) \quad \begin{aligned} T &:= \sum_{a_l \in \mathcal{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \right) \\ &\ll_{k, \mathbf{h}, \epsilon} \sum_{a_l \in \mathcal{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right). \end{aligned}$$

Indeed, if (3.11) holds, then Lemma 2.1(b) and the relation

$$\sum_{a \in \mathcal{P}_*(t'_{l-1}, t_{l-1})} \frac{\tau_{k-l+2}(a)}{a} = \prod_{t'_{l-1} < p \leq t_{l-1}} \left( 1 + \frac{k-l+2}{p} \right) \ll_k \left( \frac{\log 2t_{l-1}}{\log 2t'_{l-1}} \right)^{k-l+2}$$

imply that

$$T \ll_{k, \mathbf{h}, \epsilon} \left( \frac{\log 2t_{l-1}}{\log 2t'_{l-1}} \right)^{k-l+2} \sum_{a_l \in \mathcal{P}_*(t_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right)$$

thus completing the proof of (3.10). To prove (3.11) we decompose  $T$  into the sums

$$T_m := \sum_{\substack{a_l \in \mathcal{P}_*(t'_{l-1}, t_l) \\ a_l \in I_m}} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \right) \quad (l \leq m \leq k+1),$$

where  $I_l = (0, t_l^\delta]$ ,  $I_m = (t_{m-1}^\delta, t_m^\delta]$  if  $m \in \{l+1, \dots, k\}$  and  $I_{k+1} = (t_k^\delta, +\infty)$ . First, we estimate  $T_l$ . If  $a_l \in I_l$ , then

$$P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \geq P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \quad (l \leq i \leq k)$$

and thus we immediately deduce that

$$(3.12) \quad T_l \leq \sum_{a_l \in \mathcal{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right).$$

Next, we fix  $m \in \{l+1, \dots, k+1\}$  and bound  $T_m$ . For every  $a_l \in I_m$  we have that

$$P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \geq \begin{cases} P^+(a_l) & \text{if } l \leq i < m, \\ P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} & \text{if } m \leq i \leq k. \end{cases}$$

Moreover, the function  $a_l \rightarrow L^{(k+1)}(\mathbf{a})$  satisfies the hypothesis of Lemma 3.2 with  $C_f = k - l + 2$ , by Lemma 2.1(b). Hence

$$\begin{aligned}
T_m &\leq \left( \prod_{i=m}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right) \right) \sum_{\substack{a_l \in \mathcal{P}_*(t'_{l-1}, t_l) \\ a_l > t_{m-1}^\delta}} \frac{L^{(k+1)}(\mathbf{a})}{a_l (\log P^+(a_l))^{h_{l,l} + \cdots + h_{l,m-1}}} \\
&\ll_{k, \mathbf{h}, \epsilon} \left( \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right) \right) \left( \prod_{i=l}^{m-1} \log^{h_{l,i}} t_i \right) \\
&\quad \times \exp \left\{ -\frac{\delta \log t_{m-1}}{2 \log t_l} \right\} (\log t_l)^{-(h_{l,l} + \cdots + h_{l,m-1})} \sum_{a_l \in \mathcal{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \\
&\ll_{k, \mathbf{h}, \epsilon} \sum_{a_l \in \mathcal{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\mathbf{a})}{a_l} \prod_{i=l}^k \log^{-h_{l,i}} \left( P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \right).
\end{aligned}$$

Combining the above estimate with (3.12) shows (3.11). This completes the proof of (3.10) and hence of the lemma.  $\square$

Before we prove the upper bound in Theorem 1.7, we need one last intermediate result.

**Lemma 3.7.** *Let  $1 \leq l \leq k-1$  and  $3 \leq t_1 \leq \cdots \leq t_k$ . Then*

$$S^{(k-l+1)}(t_{l+1}, \dots, t_k) \leq (\log 2)^{-l} S^{(k+1)}(t_1, \dots, t_k)$$

and

$$S^{(k+1)}(t_1, \dots, t_k) \gg_k \log t_k.$$

*Proof.* Note that

$$\begin{aligned}
\mathcal{L}^{(k+1)}(\mathbf{a}) &\supset \bigcup_{\substack{d_1 \cdots d_l | a_1 \cdots a_l \quad (1 \leq i \leq l) \\ d_i = 1 \quad (1 \leq i \leq l)}} [\log(d_1/2), \log d_1] \times \cdots \times [\log(d_l/2), \log d_l] \\
&= [-\log 2, 0]^l \times \mathcal{L}^{(k-l+1)}(a_1 \cdots a_{l+1}, a_{l+2}, \dots, a_k)
\end{aligned}$$

and, consequently,

$$L^{(k+1)}(\mathbf{a}) \geq (\log 2)^l L^{(k-l+1)}(a_1 \cdots a_{l+1}, a_{l+2}, \dots, a_k).$$

Summing over  $\mathbf{a} \in \mathcal{P}_*^k(\mathbf{t})$  then proves the first part of the lemma.

For the second part, note that

$$S^{(k+1)}(\mathbf{a}) \geq (\log 2)^k \sum_{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{t})} \frac{1}{a_1 \cdots a_k} \asymp_k \log t_k.$$

$\square$

We are now in position to show the upper bound in Theorem 1.7. In fact, we shall prove a slightly stronger estimate, which will be useful in the proof of Theorem 1.5.

**Theorem 3.8.** Fix  $k \geq 1$ . Let  $x \geq 1$  and  $C'_k \leq y_1 \leq \dots \leq y_k$  with  $2^k y_1 \dots y_k \leq x/y_k$ . There exists a constant  $c_k$  such that

$$\frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \ll_k \left( \prod_{i=1}^k \log^{-e_{k,i}} y_i \right) \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y}) \\ a_i \leq y_i^{c_k} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k}.$$

*Proof.* Observe that it suffices to show that

$$(3.13) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \ll_k x \left( \prod_{i=1}^k \log^{-e_{k,i}} y_i \right) T,$$

where

$$T := \max\{S^{(k+1)}(\mathbf{t}) : 1 \leq t_1 \leq \dots \leq t_k, \sqrt{y_i} \leq t_i \leq 2y_i \ (1 \leq i \leq k)\}.$$

Indeed, assume for the moment that (3.13) holds. Note that

$$T \ll_k S^{(k+1)}(\mathbf{y}),$$

by Lemma 2.1(b) and inequality (1.4). Also, for every  $i \in \{1, \dots, k\}$ , we have that

$$\sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y}) \\ a_i > y_i^{c_k}}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k} \ll_k e^{-c_k/2} S^{(k+1)}(\mathbf{y}),$$

by Lemma 3.2 applied to the arithmetic function  $a_i \rightarrow L^{(k+1)}(\mathbf{a})$ . Hence, if  $c_k$  is large enough, we find that

$$T \ll_k S^{(k+1)}(\mathbf{y}) \leq 2 \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y}) \\ a_i \leq y_i^{c_k} \ (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k}$$

which, together with (3.13), completes the proof of the theorem.

In order to prove (3.13), we first reduce the counting in  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  to square-free integers. Let  $n \leq x$  be an integer counted by  $H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})$  and write  $n = ab$  with  $a$  being square-full,  $b$  square-free and  $(a, b) = 1$ . The number of  $n \leq x$  with  $a > (\log y_k)^{2k}$  is at most

$$x \sum_{\substack{a > (\log y_k)^{2k} \\ a \text{ square-full}}} \frac{1}{a} \ll \frac{x}{(\log y_k)^k}.$$

Assume now that

$$a \in I_m := \{a \in \mathbb{N} \cap ((\log y_{m-1})^{2k}, (\log y_m)^{2k}] : a \text{ square-full}\}$$

for some  $m \in \{1, \dots, k\}$ , where for the convenience of notation we have set  $y_0 = 1$ . Then we may uniquely write  $d_i = f_i e_i$ ,  $m \leq i \leq k$ , with  $f_m \dots f_k | a$  and  $e_m \dots e_k | b$ . Therefore

$$(3.14) \quad H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y}) \leq \sum_{m=1}^k \sum_{a \in I_m} \sum_{f_m \dots f_k | a} H_*^{(k-m+2)} \left( \frac{x}{a}, \left( \frac{y_m}{f_m}, \dots, \frac{y_k}{f_k} \right), 2 \left( \frac{y_m}{f_m}, \dots, \frac{y_k}{f_k} \right) \right) + O \left( \frac{x}{(\log y_k)^k} \right).$$



Fix  $m \in \{1, \dots, k\}$ ,  $a \in I_m$  and  $f_m, \dots, f_k$  with  $f_m \cdots f_k | a$ . Let  $z_m, \dots, z_k$  be the sequence  $y_m/f_m, \dots, y_k/f_k$  in increasing order and set  $\mathbf{z}' = (z_m, \dots, z_k)$ . Since  $y_m \leq \dots \leq y_k$  and

$$\frac{y_i}{f_i} \geq \frac{y_i}{a} \geq \frac{y_i}{(\log y_m)^k} \geq \sqrt{y_i} \quad (m \leq i \leq k),$$

we have that

$$(3.15) \quad \sqrt{y_i} \leq z_i \leq y_i \quad (m \leq i \leq k).$$

Next, observe that

$$(3.16) \quad H_*^{(k-m+2)}\left(\frac{x}{a}, \mathbf{z}', 2\mathbf{z}'\right) \leq \sum_{\substack{r \in \mathbb{N} \\ 2^r \leq (\log y_k)^k}} \left( H_*^{(k-m+2)}\left(\frac{x}{2^{r-1}a}, \mathbf{z}', 2\mathbf{z}'\right) - H_*^{(k-m+2)}\left(\frac{x}{2^r a}, \mathbf{z}', 2\mathbf{z}'\right) \right) + \frac{2x}{a(\log y_k)^k}.$$

For  $r$  with  $2^r \leq (\log y_k)^k$  we have that

$$\frac{x/(2^{r-1}a)}{2^{k-m+2}z_m \cdots z_k} \geq \frac{x}{2^k y_1 \cdots y_k} \frac{1}{2^r a} \geq \frac{y_k}{(\log y_k)^{3k}} \geq (2z_k)^{7/8}.$$

Thus Lemma 3.5 (applied with  $k-m+1$  in place of  $k$ ,  $x/(2^{r-1}a)$  in place of  $x$  and  $z_m, \dots, z_k$  in place of  $y_1, \dots, y_k$ ), Lemmas 3.6 and 3.7 and relation (3.15) yield

$$(3.17) \quad H_*^{(k-m+2)}\left(\frac{x}{2^{r-1}a}, \mathbf{z}', 2\mathbf{z}'\right) - H_*^{(k-m+2)}\left(\frac{x}{2^r a}, \mathbf{z}', 2\mathbf{z}'\right) \ll_k \frac{x}{2^r a} \left( \prod_{i=m}^k (\log z_i)^{-e_{k,i}} \right) S^{(k-m+2)}(2\mathbf{z}') \ll_k \frac{x}{2^r a} \left( \prod_{i=m}^k (\log y_i)^{-e_{k,i}} \right) T.$$

Since  $T \gg_k \log y_k$  by Lemma 3.7, inequalities (3.16) and (3.17) yield

$$H_*^{(k-m+2)}\left(\frac{x}{a}, \mathbf{z}', 2\mathbf{z}'\right) \ll_k \frac{x}{a} \left( \prod_{i=m}^k \log^{-e_{k,i}} y_i \right) T + \frac{x}{a(\log y_k)^k} \ll_k \frac{x}{a} \left( \prod_{i=m}^k \log^{-e_{k,i}} y_i \right) T.$$

So

$$\sum_{a \in I_m} \sum_{f_m \cdots f_k | a} H_*^{(k-m+2)}\left(\frac{x}{a}, \mathbf{z}', 2\mathbf{z}'\right) \ll_k \frac{xT}{\prod_{i=m}^k (\log y_i)^{e_{k,i}}} \sum_{a \in I_m} \frac{\tau_{k-m+2}(a)}{a} \ll_k \frac{xT}{\prod_{i=1}^k (\log y_i)^{e_{k,i}}}.$$

Inserting the above estimate into (3.14) and using the inequality  $T \gg_k \log y_k$  completes the proof of the theorem.  $\square$

**3.4. Proof of Theorem 1.1.** In this subsection we prove Theorem 1.1. Let  $3 = N_0 \leq N_1 \leq \dots \leq N_{k+1}$ . Using an inductive argument, similar to the one given in Remark 3.1, we may assume without loss of generality that  $N_1 \geq 4(C'_k)^2$ . Set  $\mathbf{N} = (N_1, \dots, N_k)$  and note that

$$(3.18) \quad A_{k+1}(N_1, \dots, N_{k+1}) \geq H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{k^2}}, \frac{\mathbf{N}}{2^k}, \frac{\mathbf{N}}{2^{k-1}}\right) \asymp_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \frac{\mathbf{N}}{2}, \mathbf{N}\right),$$

by Corollary 1.8. Also, we have that

$$(3.19) \quad A_{k+1}(N_1, \dots, N_{k+1}) \leq \sum_{\substack{1 \leq 2^{m_i} \leq N_i \\ 1 \leq i \leq k}} H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right).$$

For  $i \in \{0, 1, \dots, k\}$  let  $\mathcal{M}_i$  be the set of vectors  $\mathbf{m} \in (\mathbb{N} \cup \{0\})^k$  such that  $2^{m_j} \leq \sqrt{N_j}$  for  $i < j \leq k$  and  $\sqrt{N_i} < 2^{m_i} \leq N_i$  and set

$$T_i = \sum_{\mathbf{m} \in \mathcal{M}_i} H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right).$$

We have that

$$(3.20) \quad A_{k+1}(N_1, \dots, N_{k+1}) \leq \sum_{i=0}^k T_i,$$

by (3.19). We fix  $i \in \{0, 1, \dots, k\}$  and proceed to the estimation of  $T_i$ . Consider  $\mathbf{m} \in \mathcal{M}_i$  and let  $\mathbf{N}' = (N'_{i+1}, \dots, N'_k)$  be the vector whose coordinates are the sequence  $\{N_j/2^{m_j+1}\}_{j=i+1}^k$  in increasing order. We have that  $\sqrt{N_j} \leq 2N'_j \leq N_j$  for all  $i+1 \leq j \leq k$ . Thus

$$(3.21) \quad \begin{aligned} & H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}}, \left(\frac{N_1}{2^{m_1+1}}, \dots, \frac{N_k}{2^{m_k+1}}\right), \left(\frac{N_1}{2^{m_1}}, \dots, \frac{N_k}{2^{m_k}}\right)\right) \\ & \leq H^{(k-i+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}}, \mathbf{N}', 2\mathbf{N}'\right) \asymp_k \frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}} S^{(k-i+1)}(\mathbf{N}') \prod_{j=i+1}^k (\log N_j)^{-e_{k,j}}, \end{aligned}$$

by Theorem 1.7, with the notational convention that  $S^{(1)}(\emptyset) = 1$ . Furthermore, we have that

$$(3.22) \quad \begin{aligned} & S^{(k-i+1)}(\mathbf{N}') \leq (\log 2)^{-i} S^{(k+1)}(\sqrt{N_1}, \dots, \sqrt{N_i}, \mathbf{N}') \\ & \asymp_k S^{(k+1)}(N_1, \dots, N_k) \asymp_k \frac{H^{(k+1)}(N_1 \cdots N_{k+1}, \mathbf{N}/2, \mathbf{N})}{N_1 \cdots N_{k+1}} \prod_{j=1}^k (\log N_j)^{e_{k,j}}, \end{aligned}$$

by Lemma 3.7, Corollary 1.6 and Theorem 1.7. Combining (3.21) and (3.22) we deduce that

$$H^{(k+1)}\left(\frac{N_1 \cdots N_{k+1}}{2^{m_1 + \dots + m_k}}, \mathbf{N}', 2\mathbf{N}'\right) \ll_k \frac{H^{(k+1)}(N_1 \cdots N_{k+1}, \mathbf{N}/2, \mathbf{N})}{2^{m_1 + \dots + m_k}} (\log N_i)^{k+1}.$$

Summing the above inequality over  $\mathbf{m} \in \mathcal{M}_i$  gives us that

$$T_i \ll_k H^{(k+1)}\left(N_1 \cdots N_{k+1}, \frac{\mathbf{N}}{2}, \mathbf{N}\right) \frac{(\log N_i)^{k+1}}{\sqrt{N_i}},$$

which together with (3.18) and (3.20) completes the proof of Theorem 1.1.

#### 4. LINEAR CONSTRAINTS ON A POISSON DISTRIBUTION

A  $k$ -dimensional Poisson distribution with parameters  $z_1, \dots, z_k$  is a probability distribution on the lattice  $(\mathbb{N} \cup \{0\})^k$  that assigns to each lattice point  $(r_1, \dots, r_k)$  the probability  $\prod_{i=1}^k e^{-z_i} z_i^{r_i} / r_i!$ . Our goal in this section is to estimate the probability that lattice points obeying such a distribution lie close to a hyperplane and other related quantities. Throughout this entire section we fix positive real numbers  $\lambda_1, \dots, \lambda_k$  and we set  $\Lambda = \max_{1 \leq i \leq k} \lambda_i$ . Given  $R \geq \Lambda$ , let

$$\mathcal{H}^k(R) = \left\{ (r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : R - \Lambda < \sum_{i=1}^k \lambda_i r_i \leq R \right\}$$

$$\mathcal{H}_-^k(R) = \left\{ (r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : \sum_{i=1}^k \lambda_i r_i \leq R \right\}$$

and

$$\mathcal{H}_+^k(R) = \left\{ (r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : \sum_{i=1}^k \lambda_i r_i \geq R \right\}.$$

Also, define the number  $\alpha(R) = \alpha(R; k, \mathbf{z}, \boldsymbol{\lambda})$  implicitly via the equation

$$\sum_{i=1}^k \lambda_i e^{\alpha(R)\lambda_i} z_i = R$$

and set

$$\mathcal{H}^k(R, \delta) = \left\{ \mathbf{r} \in \mathcal{H}^k(R) : |r_i - e^{\alpha(R)\lambda_i} z_i| \leq \frac{\Lambda}{\lambda_i} \max \left\{ k, \delta \sqrt{e^{\alpha(R)\lambda_i} z_i} \right\} \quad (1 \leq i \leq k) \right\}.$$

*Remark 4.1.* The motivation for the definition of  $\alpha(R)$  may be briefly summarized as follows: By Stirling's formula, we have that

$$(4.1) \quad \prod_{i=1}^k \frac{z_i^{r_i}}{r_i!} \sim_k \prod_{i=1}^k \frac{1}{\sqrt{2\pi r_i}} \left( \frac{z_i e}{r_i} \right)^{r_i}.$$

Using Lagrange multipliers, we see that when  $\mathbf{r}$  ranges over  $\mathcal{H}^k(R)$ , the maximum of the right hand side in (4.1) occurs when  $r_i = e^{\alpha(R)\lambda_i} z_i + O_{k,\boldsymbol{\lambda}}(1)$  for all  $i \in \{1, \dots, k\}$ .

**Lemma 4.1.** *Let  $k \in \mathbb{N}$ ,  $0 < \delta \leq 1$ ,  $z_1, \dots, z_k \geq 1$  and  $\lambda_1, \dots, \lambda_k > 0$ . There is a constant  $c = c(k, \boldsymbol{\lambda})$  such that:*

(1) *If  $R \geq \max\{\Lambda, \delta(z_1 + \dots + z_k)\}$ , then*

$$\text{Prob}(\mathcal{H}^k(R, \delta)) \gg_{k,\boldsymbol{\lambda},\delta} \frac{e^{-c|\alpha(R)|}}{\sqrt{R}} \prod_{i=1}^k \exp\{-Q(e^{\alpha(R)\lambda_i} z_i)\}.$$

(2) If  $R \geq \Lambda$ , then

$$\mathbf{Prob}(\mathcal{H}^k(R)) \ll_{k,\lambda} \frac{e^{c|\alpha(R)|}}{\sqrt{R}} \prod_{i=1}^k \exp\{-Q(e^{\alpha(R)\lambda_i})z_i\}.$$

*Proof.* By Stirling's formula, we have that

$$(4.2) \quad \prod_{i=1}^k e^{-z_i} \frac{z_i^{r_i}}{r_i!} \asymp_k \left( \prod_{i=1}^k \frac{\sqrt{r_i+1}}{z_i} \right) e^{F(\mathbf{r})},$$

where

$$(4.3) \quad F(\mathbf{r}) = -(z_1 + \cdots + z_k) + \sum_{i=1}^k (r_i + 1) \log \frac{z_i e}{r_i + 1}.$$

Set  $r_i^* = e^{\lambda_i \alpha(R)} z_i - 1$  for  $i \in \{1, \dots, k\}$ . Without loss of generality, assume that  $r_k^* + 1 = \max_{1 \leq i \leq k} (r_i^* + 1)$ , so that

$$(4.4) \quad r_k^* + 1 \asymp_{k,\lambda} R.$$

In order to prove part (a) of the lemma, we shall employ quadratic approximation to  $F(\mathbf{r})$  around the point  $\mathbf{r}^*$ . However, for part (b) we need to be more careful: we shall reparametrize the set  $\mathcal{H}^k(R)$  first and then use the saddle point method. We give the details of the proof below.

(a) Since  $R \geq \delta(z_1 + \cdots + z_k)$ , then (4.4) yields

$$e^{\lambda_k \alpha(R)} z_k \gg_{k,\lambda} R \geq \delta z_k$$

and thus  $\alpha(R) \geq -C$  for some constant  $C = C(k, \lambda, \delta)$ . In turn, this implies that  $r_i^* + 1 \gg_{k,\lambda,\delta} z_i \geq 1$  for all  $i \in \{1, \dots, k\}$ . By Taylor's theorem, for every  $\mathbf{r} \in \mathcal{H}^k(R, \delta)$  there is a vector  $\boldsymbol{\xi} \in \mathbb{R}^k$  that lies on the line segment connecting  $\mathbf{r}$  and  $\mathbf{r}^*$  and satisfies

$$(4.5) \quad \begin{aligned} F(\mathbf{r}) &= F(\mathbf{r}^*) + \sum_{i=1}^k \frac{\partial F(\mathbf{r}^*)}{\partial x_i} (r_i - r_i^*) + \frac{1}{2} \sum_{1 \leq i, j \leq k} \frac{\partial^2 F(\boldsymbol{\xi})}{\partial x_i \partial x_j} (r_i - r_i^*)(r_j - r_j^*) \\ &= F(\mathbf{r}^*) - \sum_{i=1}^k \frac{(r_i - r_i^*)^2}{2\xi_i + 2} + O_{k,\lambda}(|\alpha(R)|) = F(\mathbf{r}^*) + O_{k,\lambda}(1 + |\alpha(R)|). \end{aligned}$$

Since we also have that

$$\begin{aligned} |\mathcal{H}^k(R, \delta)| &\geq \left| \left\{ \mathbf{r} \in \mathcal{H}^k(R) : |r_i - r_i^* - 1| \leq \frac{\Lambda}{k\lambda_i} \max \left\{ k, \delta \sqrt{r_i^* + 1} - 1 \right\} \quad (1 \leq i \leq k-1) \right\} \right| \\ &\asymp_{k,\lambda,\delta} \prod_{i=1}^{k-1} \sqrt{r_i^* + 1} \asymp_{k,\lambda} \sqrt{\frac{(r_1^* + 1) \cdots (r_{k-1}^* + 1)}{R}}, \end{aligned}$$

the desired lower bound on  $\mathbf{Prob}(\mathcal{H}^k(R, \delta))$  follows.

(b) Let

$$\mathcal{R} = \{\tilde{\mathbf{r}} = (r_1, \dots, r_{k-1}) \in (\mathbb{N} \cup \{0\})^{k-1} : \lambda_1 r_1 + \cdots + \lambda_{k-1} r_{k-1} \leq R\}$$

and, for  $\tilde{\mathbf{r}} \in \mathcal{R}$ , set

$$f(\tilde{\mathbf{r}}) = \frac{1}{\lambda_k} \left( R - \sum_{i=1}^{k-1} \lambda_i r_i \right) \quad \text{and} \quad G(\tilde{\mathbf{r}}) = F(\tilde{\mathbf{r}}, f(\tilde{\mathbf{r}})),$$

where  $F$  is defined by (4.3). Given  $\tilde{\mathbf{r}} \in \mathcal{R}$ , there is a positive but bounded number of integers  $r_k$  such that  $(r_1, \dots, r_k) \in \mathcal{H}^k(R)$ : Indeed, we have that  $(r_1, \dots, r_k) \in \mathcal{H}^k(R)$  if, and only if,

$$(4.6) \quad r_k \geq 0 \quad \text{and} \quad f(\tilde{\mathbf{r}}) - \Lambda/\lambda_k < r_k \leq f(\tilde{\mathbf{r}}).$$

Also, relation (4.6) and the Mean Value Theorem imply that there is some  $\xi \in (r_k+1, f(\tilde{\mathbf{r}})+1)$  such that

$$(r_k + 1) \log \frac{z_k e}{r_k + 1} - (f(\tilde{\mathbf{r}}) + 1) \log \frac{z_k e}{f(\tilde{\mathbf{r}}) + 1} = (f(\tilde{\mathbf{r}}) - r_k) \log \frac{\xi}{z_k}.$$

We have that

$$\log \frac{\xi}{z_k} \leq \log \frac{R/\lambda_k + 1}{z_k} = \log \frac{r_k^* + 1}{z_k} + O_{k,\lambda}(1) \leq \lambda_k |\alpha(R)| + O_{k,\lambda}(1),$$

by (4.4). So (4.2) yields that

$$\mathbf{Prob}(\mathcal{H}^k(R)) \ll_{k,\lambda} e^{\Lambda|\alpha(R)|} \sum_{\tilde{\mathbf{r}} \in \mathcal{R}} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) \frac{\sqrt{f(\tilde{\mathbf{r}}) + 1}}{z_k} e^{G(\tilde{\mathbf{r}})}.$$

Since we also have that  $f(\tilde{\mathbf{r}}) + 1 \leq R/\lambda_k + 1 \asymp_{k,\lambda} r_k^* + 1$ , by (4.4), we deduce that

$$(4.7) \quad \mathbf{Prob}(\mathcal{H}^k(R)) \ll_{k,\lambda} \frac{e^{O_{k,\lambda}(|\alpha(R)|)}}{\sqrt{R}} \sum_{\tilde{\mathbf{r}} \in \mathcal{R}} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})}.$$

In order to estimate the right hand side of (4.7), we shall use quadratic approximation to  $G(\tilde{\mathbf{r}})$  around the point  $\tilde{\mathbf{r}}^* = (r_1^*, \dots, r_{k-1}^*)$ . We have that

$$\frac{\partial G(\tilde{\mathbf{r}}^*)}{\partial x_i} = \log \frac{z_i}{r_i^* + 1} + \frac{\lambda_i}{\lambda_k} \log \frac{f(\tilde{\mathbf{r}}^*) + 1}{z_k} = \frac{\lambda_i}{\lambda_k} \log \frac{f(\tilde{\mathbf{r}}^*) + 1}{r_k^* + 1} \ll_{k,\lambda} \frac{1}{R} \quad (1 \leq i \leq k-1),$$

by (4.4) and (4.6). Also,

$$\frac{\partial^2 G}{\partial x_i \partial x_j}(\tilde{\mathbf{r}}) = -\frac{\delta_{i,j}}{r_i + 1} - \frac{\lambda_i \lambda_j}{\lambda_k^2 (f(\tilde{\mathbf{r}}) + 1)} \quad (1 \leq i, j \leq k-1),$$

where  $\delta_{i,j}$  is the standard Kronecker symbol. So for every  $\tilde{\mathbf{r}} \in \mathcal{R}$  there is a vector  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{k-1})$  that lies on the line segment connecting  $\tilde{\mathbf{r}}$  and  $\tilde{\mathbf{r}}^*$  and satisfies

$$(4.8) \quad \begin{aligned} G(\tilde{\mathbf{r}}) &= G(\tilde{\mathbf{r}}^*) + O_{k,\lambda} \left( \frac{1}{R} \sum_{i=1}^{k-1} |r_i - r_i^*| \right) - \sum_{i=1}^{k-1} \frac{(r_i - r_i^*)^2}{2(\xi_i + 1)} - \frac{1}{2} \left( \sum_{i=1}^{k-1} \frac{\lambda_i (r_i - r_i^*)}{\lambda_k \sqrt{f(\boldsymbol{\xi}) + 1}} \right)^2 \\ &= G(\tilde{\mathbf{r}}^*) + O_{k,\lambda}(1) - \sum_{i=1}^{k-1} \frac{(r_i - r_i^*)^2}{2\xi_i + 2} - \frac{(f(\tilde{\mathbf{r}}) - f(\tilde{\mathbf{r}}^*))^2}{2f(\boldsymbol{\xi}) + 2}. \end{aligned}$$

Next, we split the set  $\mathcal{R}$  into certain subsets. Let

$$\mathcal{R}_1 = \{\tilde{\mathbf{r}} \in \mathcal{R} : f(\tilde{\mathbf{r}}) + 1 > (1 + \eta)(r_k^* + 1) - \Lambda/\lambda_k\},$$

$$\mathcal{R}_2 = \{\tilde{\mathbf{r}} \in \mathcal{R} \setminus \mathcal{R}_1 : r_i \leq 3r_i^* + 4 \ (1 \leq i \leq k-1)\},$$

where  $\eta = -1 + 2^{\lambda_k/\Lambda} > 0$ , and for  $I \subset \{1, \dots, k-1\}$  set

$$\mathcal{R}_3(I) = \{\tilde{\mathbf{r}} \in \mathcal{R} \setminus \mathcal{R}_1 : r_i > 3r_i^* + 4 \ (i \in I), r_i \leq 3r_i^* + 4 \ (i \notin I, 1 \leq i \leq k-1)\}.$$

If  $\tilde{\mathbf{r}} \in \mathcal{R}_1$ , then (4.8) implies that

$$G(\tilde{\mathbf{r}}) \leq G(\tilde{\mathbf{r}}^*) + O_{k,\lambda}(1) - \frac{(\eta r_k^* - O_{k,\lambda}(1))^2}{2R/\lambda_k} \leq G(\tilde{\mathbf{r}}^*) + O_{k,\lambda}(1) - c_0 R$$

for some positive constant  $c_0 = c_0(k, \lambda)$ , by (4.4). Therefore

$$(4.9) \quad \sum_{\tilde{\mathbf{r}} \in \mathcal{R}_1} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})} \ll_{k,\lambda} R^{3(k-1)/2} e^{G(\tilde{\mathbf{r}}^*) - c_0 R} \ll_{k,\lambda} \frac{e^{G(\tilde{\mathbf{r}}^*)}}{\sqrt{R}}.$$

Next, if  $\tilde{\mathbf{r}} \in \mathcal{R}_2$ , then for any  $\xi$  that lies on the line segment connecting  $\tilde{\mathbf{r}}$  and  $\tilde{\mathbf{r}}^*$  we have that  $\xi_i \leq 3r_i^* + 4$ . Consequently,

$$G(\tilde{\mathbf{r}}) \leq G(\tilde{\mathbf{r}}^*) + O_{k,\lambda}(1) - \sum_{i=1}^{k-1} \frac{(r_i - r_i^*)^2}{6r_i^* + 10},$$

by (4.8). So we deduce that

$$(4.10) \quad \sum_{\tilde{\mathbf{r}} \in \mathcal{R}_2} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})} \ll_{k,\lambda} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i^* + 2}}{z_i} \right) e^{G(\tilde{\mathbf{r}}^*)} \prod_{i=1}^{k-1} \sqrt{r_i^* + 2} = e^{O_{k,\lambda}(1 + |\alpha(R)|) + G(\tilde{\mathbf{r}}^*)},$$

where we used (4.4). Lastly, fix some non-empty set  $I \subset \{1, \dots, k-1\}$  and  $i \in I$  and consider  $\tilde{\mathbf{r}} \in \mathcal{R}_3(I)$ . Set

$$\tilde{\mathbf{r}}_i = (r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_k).$$

Then for every vector  $\mathbf{s}$  that lies in the line segment connecting  $\tilde{\mathbf{r}}$  and  $\tilde{\mathbf{r}}_i$  we have that

$$\frac{\partial G}{\partial x_i}(\mathbf{s}) \leq \log \frac{z_i}{r_i - 1} + \frac{\lambda_i}{\lambda_k} \log \frac{f(\tilde{\mathbf{r}}_i) + 1}{z_k} \leq \log \frac{z_i}{3r_i^* + 3} + \frac{\lambda_i}{\lambda_k} \log \frac{(1 + \eta)(r_k^* + 1)}{z_k} \leq -\log \frac{3}{2}.$$

So by the Mean Value Theorem we find that  $e^{G(\tilde{\mathbf{r}})} \leq \frac{2}{3} e^{G(\tilde{\mathbf{r}}_i)}$  and, consequently,

$$\sum_{\tilde{\mathbf{r}} \in \mathcal{R}_3(I)} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})} \ll_{k,\lambda} \sum_{\tilde{\mathbf{r}} \in \mathcal{R}_3(I \setminus \{i\}) \cup \mathcal{R}_1} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})}.$$

Iterating the above inequality yields that

$$\sum_{\tilde{\mathbf{r}} \in \mathcal{R}_3(I)} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})} \ll_{k,\lambda} \sum_{\tilde{\mathbf{r}} \in \mathcal{R}_1 \cup \mathcal{R}_2} \left( \prod_{i=1}^{k-1} \frac{\sqrt{r_i + 1}}{z_i} \right) e^{G(\tilde{\mathbf{r}})},$$

since  $\mathcal{R}_3(\emptyset) = \mathcal{R}_2$ . Combining the above estimate with relations (4.9), (4.10) and (4.7) shows that

$$\mathbf{Prob}(\mathcal{H}^k(R)) \leq \frac{e^{O_{k,\lambda}(1+|\alpha(R)|)+G(\tilde{\mathbf{r}}^*)}}{\sqrt{R}}.$$

To complete the proof of the lemma, note that

$$|G(\tilde{\mathbf{r}}^*) - F(\mathbf{r}^*)| = |F(\tilde{\mathbf{r}}^*, f(\tilde{\mathbf{r}}^*)) - F(\tilde{\mathbf{r}}^*, r_k^*)| \ll_{k,\lambda} 1 + |\alpha(R)|$$

and

$$F(\mathbf{r}^*) = - \sum_{i=1}^k Q(e^{\lambda_i \alpha(R)}) z_i.$$

□

Finally, as a consequence of Lemma 4.1, we have the following estimates.

**Lemma 4.2.** *Let  $k \in \mathbb{N}$ ,  $C \geq 0$ ,  $z_1, \dots, z_k \geq 1$ ,  $\lambda_1, \dots, \lambda_k > 0$  and  $\mu_1, \dots, \mu_k > 0$  such that  $Z = \mu_1 z_1 + \dots + \mu_k z_k \geq \Lambda$ .*

(a) *If  $\lambda_i < \mu_i$  for all  $i \in \{1, \dots, k\}$ , then*

$$\sum_{\mathbf{r} \in \mathcal{H}_+^k(Z)} \left( 1 + \sum_{i=1}^k \lambda_i r_i - Z \right)^C \prod_{i=1}^k \frac{e^{-z_i} z_i^{r_i}}{r_i!} \ll_{k,\lambda,\mu,C} \mathbf{Prob}(\mathcal{H}^k(Z)).$$

(b) *If  $\log(\mu_i/\lambda_i) < \lambda_i$  for all  $i \in \{1, \dots, k\}$ , then*

$$\sum_{\mathbf{r} \in \mathcal{H}_-^k(Z)} \left( 1 + Z - \sum_{i=1}^k \lambda_i r_i \right)^C \prod_{i=1}^k \frac{e^{-z_i} (e^{\lambda_i} z_i)^{r_i}}{r_i!} \ll_{k,\lambda,\mu,C} e^Z \mathbf{Prob}(\mathcal{H}^k(Z)).$$

*Proof.* (a) Let  $S_+$  be the sum in question. If we set

$$G(R) = - \sum_{i=1}^k Q(e^{\alpha(R)\lambda_i}) z_i = -R \alpha(R) + \sum_{i=1}^k (e^{\alpha(R)\lambda_i} - 1) z_i,$$

then Lemma 4.1(b) implies that

$$(4.11) \quad S_+ \ll_{k,C,\lambda,\mu} \sum_{n=0}^{\infty} (1+n)^C \frac{\exp\{c|\alpha(Z+n\Lambda)| + G(Z+n\Lambda)\}}{\sqrt{Z+n\Lambda}}.$$

Differentiating implicitly the defining equation of  $\alpha(R)$ , we find that there are positive constants  $c_1 = c_1(k, \lambda)$  and  $c_2 = c_2(k, \lambda)$  such that

$$\frac{c_1}{R} \leq \alpha'(R) = \left( \sum_{i=1}^k \lambda_i^2 e^{\alpha(R)\lambda_i} z_i \right)^{-1} \leq \frac{c_2}{R} \quad (R \geq \Lambda).$$

Also, we have that

$$(4.12) \quad \alpha(Z) \geq \min_{1 \leq i \leq k} \frac{1}{\lambda_i} \log \left( \frac{\mu_i}{\lambda_i} \right) > 0,$$

by the definition of  $\alpha(Z)$  and our assumption that  $\lambda_i < \mu_i$  for all  $i$ . So

$$G'(R) = -\alpha(R) \leq -\alpha(Z) < 0 \quad (R \geq Z).$$

Combining the above remarks, we see that the summands in the right hand side of (4.11) decay exponentially. Hence

$$S_+ \ll_{k,C,\lambda,\mu} \frac{\exp\{c\alpha(Z) + G(Z)\}}{\sqrt{Z}},$$

which together with Lemma 4.1(a) implies that

$$S_+ \ll_{k,C,\lambda,\mu} e^{2c\alpha(Z)} \mathbf{Prob}(\mathcal{H}^k(Z)).$$

To complete the proof, note that

$$(4.13) \quad \alpha(Z) \leq \max_{1 \leq i \leq k} \frac{1}{\lambda_i} \log \left( \frac{\mu_i}{\lambda_i} \right) \ll_{k,\lambda,\mu} 1,$$

by the definition of  $\alpha(Z)$ .

(b) We argue as in part (a). Let  $S_-$  be the sum we want to estimate. Then

$$S_- \ll_{k,C,\lambda,\mu} \sum_{0 \leq n \leq Z/\Lambda - 1} (1+n)^C \frac{\exp\{c|\alpha(Z - n\Lambda)| + H(Z - n\Lambda)\}}{\sqrt{Z - n\Lambda}},$$

where  $H(R) = R + G(R)$ , by Lemma 4.1(b). We have that

$$H'(R) = 1 - \alpha(R) \geq 1 - \alpha(Z) > 0 \quad (R \geq Z),$$

by the first inequality in (4.13) and our assumption that  $\log(\mu_i/\lambda_i) < \lambda_i$  for all  $i$ . Thus

$$S_- \ll_{k,C,\lambda,\mu} \frac{\exp\{c|\alpha(Z)| + H(Z)\}}{\sqrt{Z}} \ll_{k,\lambda,\mu} e^{Z+2c|\alpha(Z)|} \mathbf{Prob}(\mathcal{H}^k(Z)) \ll_{k,\lambda} e^Z \mathbf{Prob}(\mathcal{H}^k(Z)),$$

by Lemma 4.1(a) and relations (4.12) and (4.13). This completes the proof of the lemma.  $\square$

## 5. THE UPPER BOUND IN THEOREM 1.5

**5.1. Outline of the proof.** In this subsection we give the key steps of the proof of the upper bound in Theorem 1.5 with most of the technical details omitted. Observe that, in view of Corollary 1.8, we may assume that the numbers  $\ell_1, \dots, \ell_k$  are sufficiently large. Our starting point is Theorem 3.8. We break the sum

$$S^{(k+1)}(\mathbf{a}) = \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y}) \\ a_i \leq y_i^{c_k} (1 \leq i \leq k)}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k}$$

into pieces according to the number of prime factors of the variables  $a_1, \dots, a_k$ . More precisely, set  $\omega_k(a) = |\{p|n : p > k\}|$  and

$$S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) = \sum_{\substack{\mathbf{a} \in \mathcal{P}_*^k(\mathbf{y}) \\ \omega_k(a_i) = r_i, a_i \leq y_i^{c_k} \\ 1 \leq i \leq k}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \quad (\mathbf{r} \in (\mathbb{N} \cup \{0\})^k).$$



Also, for each fixed  $i \in \{1, \dots, k\}$  define a sequence of prime numbers  $\lambda_{i,1}, \lambda_{i,2}, \dots$ , as follows. Set  $\rho_m = (m+1)^{1/m}$  for  $m \in \mathbb{N}$ ,  $\lambda_{i,0} = \max\{k, y_{i-1}\}$  and define inductively  $\lambda_{i,j}$  as the largest element of the set  $\{p \text{ prime} : \lambda_{i,0} < p \leq y_i\}$  such that

$$(5.1) \quad \sum_{\lambda_{i,j-1} < p \leq \lambda_{i,j}} \frac{1}{p} \leq \log \rho_{k-i+1}.$$

Notice that the sequence  $\{\lambda_{i,j}\}_{j \in \mathbb{N}}$  eventually becomes constant. Let  $v_i$  be the smallest integer satisfying  $\lambda_{i,v_i} = \lambda_{i,v_i+1}$ . Set

$$D_{i,j} = \{p \text{ prime} : \lambda_{i,j-1} < p \leq \lambda_{i,j}\} \quad (1 \leq i \leq k, 1 \leq j \leq v_i)$$

and observe that

$$(5.2) \quad \bigcup_{j=1}^{v_i} D_{i,j} = \{p \text{ prime} : \max\{y_{i-1}, k\} < p \leq y_i\} \quad (1 \leq i \leq k).$$

Also, we have the following estimate.

**Lemma 5.1.** *There exists some positive number  $L_k$  such that*

$$(\rho_{k-i+1})^{j-L_k} \leq \frac{\log \lambda_{i,j}}{\log y_{i-1}} \leq (\rho_{k-i+1})^{j+L_k} \quad (1 \leq i \leq k, 1 \leq j \leq v_i).$$

Consequently, we have that

$$v_i = \frac{\ell_i}{\log \rho_{k-i+1}} + O_k(1) \quad (1 \leq i \leq k).$$

*Proof.* The proof is similar to the proof of Lemma 4.6 in [Fo08b] and Lemma 3.4 in [K10a].  $\square$

Set  $\mathbf{v} = (v_1, \dots, v_k)$ ,

$$\Delta_r = \{(\xi_1, \dots, \xi_r) \in \mathbb{R}^r : 0 \leq \xi_1 \leq \dots \leq \xi_r \leq 1\}$$

and for  $i \in \{1, \dots, k\}$  and  $\boldsymbol{\xi}_i = (\xi_{i,1}, \dots, \xi_{i,r_i}) \in \Delta_{r_i}$  define

$$F_i(\boldsymbol{\xi}_i) = \left( \min_{0 \leq j \leq r_i} \rho_{k-i+1}^{-j} (1 + \rho_{k-i+1}^{v_i \xi_{i,1}} + \dots + \rho_{k-i+1}^{v_i \xi_{i,j}}) \right)^{k-i+1}.$$

We shall bound  $S_{\mathbf{r}}^{(k+1)}(\mathbf{y})$  in terms of

$$U_{\mathbf{r}}^{(k+1)}(\mathbf{v}) = \int \dots \int_{\substack{\boldsymbol{\xi}_i \in \Delta_{r_i} \\ F_i(\boldsymbol{\xi}_i) \leq C_k(k-i+2)^{v_i-r_i} \\ 1 \leq i \leq k}} \min_{1 \leq i \leq k} \left\{ F_i(\boldsymbol{\xi}_i) \prod_{m=1}^{i-1} (k-m+2)^{v_m-r_m} \right\} d\boldsymbol{\xi}_1 \dots d\boldsymbol{\xi}_k,$$

where  $C_k$  is a sufficiently large constant.

**Lemma 5.2.** *If  $y_1$  is large enough, then*

$$S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) \ll_k U_{\mathbf{r}}^{(k+1)}(\mathbf{v}) \prod_{i=1}^k (v_i(k-i+2) \log \rho_{k-i+1})^{r_i}.$$

Lemma 5.2 will be proven in Subsection 5.2. Next, we give an upper bound on  $U_{\mathbf{r}}^{(k+1)}(\mathbf{v})$ , but first we need to introduce some notation. For  $\mathbf{r} \in (\mathbb{N} \cup \{0\})^k$ ,  $1 \leq i \leq k+1$  and  $1 \leq j \leq k+1$  set

$$B_{i,j} = \begin{cases} -\sum_{m=j}^{i-1} (r_m - v_m) \log(k - m + 2) & \text{if } 1 \leq j < i, \\ 0 & \text{if } j = i, \\ \sum_{m=i}^{j-1} (r_m - v_m) \log(k - m + 2) & \text{if } i < j. \end{cases}$$

Observe that

$$(5.3) \quad B_{i,m} + B_{m,j} = B_{i,j} \quad (1 \leq i, m, j \leq k+1).$$

For  $j \in \{1, \dots, k+1\}$  set

$$\mathcal{R}_j = \{\mathbf{r} \in (\mathbb{N} \cup \{0\})^k : B_{i,j} \geq 0 \ (1 \leq i \leq k+1)\}.$$

Then

$$(5.4) \quad \bigcup_{j=1}^{k+1} \mathcal{R}_j = (\mathbb{N} \cup \{0\})^k.$$

Indeed, for every  $\mathbf{r} \in (\mathbb{N} \cup \{0\})^k$  there is some  $j \in \{1, \dots, k+1\}$  such that  $B_{1,j} \geq B_{1,i}$  for all  $i \in \{1, \dots, k+1\}$ . So  $\mathbf{r} \in \mathcal{R}_j$ , by (5.3).

The following estimate will be shown in Subsection 5.2.

**Lemma 5.3.** *Let  $j \in \{1, \dots, k+1\}$  and  $\mathbf{r} \in \mathcal{R}_j$ . Then*

$$U_{\mathbf{r}}^{(k+1)}(\mathbf{v}) \ll_k \min \left\{ 1, \frac{(1 + B_{i_0,j})(1 + B_{i_0+1,j})}{r_{i_0} + 1} \right\} \frac{\prod_{m=1}^{j-1} (k - m + 2)^{v_m - r_m}}{r_1! \cdots r_k!}.$$

By Lemma 5.3 and the results of Section 4, we obtain the following estimate, which will be proven in Subsection 5.2.

**Lemma 5.4.** *We have that*

$$\sum_{\mathbf{r} \in (\mathbb{N} \cup \{0\})^k} S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) \ll_k \beta \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{k-i+2-Q((k-i+2)^\alpha)}.$$

The upper bound in Theorem 1.5 now follows immediately by Theorem 3.8 and Lemma 5.4.

**5.2. Completion of the proof.** In this subsection we give the proofs of Lemmas 5.2, 5.3 and 5.4.

*Proof of Lemma 5.2.* Let  $a_1 = a'_1 p_{1,1} \cdots p_{1,r_1} \leq y_1^c$  with  $a'_1 \in \mathcal{P}_*(1, k)$  and  $k < p_{1,1} < \cdots < p_{1,r_1} \leq y_1$ . Also, for  $m \in \{2, \dots, k\}$  let  $a_m = p_{m,1} \cdots p_{m,r_m} \leq y_m^c$  with  $y_{m-1} < p_{m,1} < \cdots < p_{m,r_m}$ . For each  $m \in \{1, \dots, k\}$  let  $b_m = p_{m,1} \cdots p_{m,r_m}$ . Also, for  $1 \leq m \leq k$  and  $1 \leq i \leq r_m$

define  $n_{m,i} \in \{1, \dots, v_m\}$  by  $p_{m,i} \in D_{m,n_{m,i}}$  and put  $\mathbf{n}_m = (n_{m,1}, \dots, n_{m,r_m})$ . For every  $i \in \{1, \dots, k\}$  Lemma 2.1(b) implies that

$$\begin{aligned}
 L^{(k+1)}(\mathbf{a}) &\leq \tau_{k+1}(a'_1, \underbrace{1, \dots, 1}_{i-1 \text{ times}}, b_{i+1}, \dots, b_k) L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\
 (5.5) \quad &= \tau_{k+1}(a'_1) \left( \prod_{m=i+1}^k (k-m+2)^{r_m} \right) L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}).
 \end{aligned}$$

Moreover, Lemmas 2.1 and 5.1 together with our assumption that  $a_i \leq y_i^{c_k}$  for  $1 \leq i \leq k$  imply that for every  $j \in \{0, 1, \dots, r_i\}$  we have

$$\begin{aligned}
 &L^{(k+1)}(b_1, \dots, b_i, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\
 &\leq (k-i+2)^{r_i-j} L^{(k+1)}(b_1, \dots, b_{i-1}, p_{i,1} \cdots p_{i,j}, \underbrace{1, \dots, 1}_{k-i \text{ times}}) \\
 (5.6) \quad &\leq (k-i+2)^{r_i-j} \left( \prod_{m=1}^{i-1} \log(2b_1 \cdots b_m) \right) (\log(2b_1 \cdots b_{i-1}) + \log(p_{i,1} \cdots p_{i,j}))^{k-i+1} \\
 &\ll_k (k-i+2)^{r_i-j} \left( \prod_{m=1}^{i-1} \log y_m \right) (\log y_{i-1} (1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,j}}))^{k-i+1} \\
 &\asymp_k (k-i+2)^{r_i} \left( \prod_{m=1}^{i-1} (k-m+2)^{v_m} \right) (\rho_{k-i+1}^{-j} (1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,j}}))^{k-i+1}
 \end{aligned}$$

So if we set

$$G_i(\mathbf{n}_i) = \left( \min_{0 \leq j \leq r_i} \rho_{k-i+1}^{-j} (1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,j}}) \right)^{k-i+1}$$

and

$$G(\mathbf{n}_1, \dots, \mathbf{n}_k) = \min_{1 \leq i \leq k} \left\{ G_i(\mathbf{n}_i) \prod_{m=1}^{i-1} (k-m+2)^{v_m-r_m} \right\},$$

then we find that

$$L^{(k+1)}(\mathbf{a}) \ll_k \tau_{k+1}(a'_1) G(\mathbf{n}_1, \dots, \mathbf{n}_k) \prod_{i=1}^k (k-i+2)^{r_i}.$$

Next, note that

$$\begin{aligned}
 (5.7) \quad G_i(\mathbf{n}_i) &\leq (k-i+2)^{-r_i} (1 + \rho_{k-i+1}^{n_{i,1}} + \cdots + \rho_{k-i+1}^{n_{i,r_i}})^{k-i+1} \\
 &\asymp_k (k-i+2)^{-r_i} \left( \frac{\log 2a_i}{\log y_{i-1}} \right)^{k-i+1} \ll_k (k-i+2)^{v_i-r_i},
 \end{aligned}$$

by Lemma 5.1 and our assumption that  $a_i \leq y_i^{c_k}$ . Also,

$$\sum_{a'_1 \in \mathcal{P}_*(1,k)} \frac{\tau_{k+1}(a'_1)}{a'_1} \ll_k 1.$$

So if  $\mathcal{N}$  denotes the set of  $k$ -tuples  $\mathbf{n} = (\mathbf{n}_1, \dots, \mathbf{n}_k)$  satisfying  $1 \leq n_{m,1} \leq \dots \leq n_{m,r_m} \leq v_m$  for  $1 \leq m \leq k$  and inequality (5.7), then

$$(5.8) \quad S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) \ll_k \sum_{\mathbf{n} \in \mathcal{N}} G(\mathbf{n}) \prod_{i=1}^k \left( (k-i+2)^{r_i} \sum_{\substack{p_{i,1} < \dots < p_{i,r_i} \\ p_{i,j} \in D_{i,n_{i,j}} \\ 1 \leq j \leq r_i}} \frac{1}{p_{i,1} \cdots p_{i,r_i}} \right).$$

Fix  $i \in \{1, \dots, k\}$ . Let  $g_{i,s} = |\{1 \leq j \leq r_i : n_{i,j} = s\}|$  for  $s \in \{1, \dots, v_i\}$ . By (5.1), the sum over  $p_{i,1}, \dots, p_{i,r_i}$  in (5.8) is at most

$$(5.9) \quad \prod_{s=1}^{v_i} \frac{1}{g_{i,s}!} \left( \sum_{p \in D_{i,s}} \frac{1}{p} \right)^{g_{i,s}} \leq \frac{(\log \rho_{k-i+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!} = (v_i \log \rho_{k-i+1})^{r_i} \text{Vol}(I(\mathbf{n}_i)),$$

where

$$I(\mathbf{n}_i) := \{\boldsymbol{\xi}_i \in \Delta_{r_i} : n_{i,j} - 1 \leq v_i \xi_{i,j} < n_{i,j} \ (1 \leq j \leq r_i)\}.$$

By (5.8) and (5.9) we deduce that

$$(5.10) \quad S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) \ll_k \left( \prod_{i=1}^k (v_i (k-i+2) \log \rho_{k-i+1})^{r_i} \right) \sum_{\mathbf{n} \in \mathcal{N}} G(\mathbf{n}) \text{Vol}(I(\mathbf{n}_1) \times \cdots \times I(\mathbf{n}_k)).$$

Finally, note that the definition of  $I(\mathbf{n}_i)$  and (5.7) imply that

$$G_i(\mathbf{n}_i) \leq (k-i+2) F_i(\boldsymbol{\xi}_i) \leq (k-i+2) G_i(\mathbf{n}_i) \leq C_k (k-i+2)^{v_i - r_i + 1} \quad (\boldsymbol{\xi}_i \in I(\mathbf{n}_i))$$

for some sufficiently large constant  $C_k$  and, consequently,

$$\sum_{\mathbf{n} \in \mathcal{N}} G(\mathbf{n}) \text{Vol}(I(\mathbf{n}_1) \times \cdots \times I(\mathbf{n}_k)) \ll_k U_{\mathbf{r}}^{(k+1)}(\mathbf{v}).$$

Inserting the above estimate into (5.10) completes the proof of the lemma.  $\square$

Our next goal is to show Lemma 5.3. First, we state an auxiliary result.

**Lemma 5.5.** *Let  $\mu > 1$ ,  $A \geq 0$ ,  $r, v \in \mathbb{N}$  and  $\gamma \geq 0$ . Consider the set  $\mathcal{T}_{\mu}(r, v, \gamma)$  of all vectors  $(\xi_1, \dots, \xi_r) \in \Delta_r$  such that  $\mu^{v\xi_1} + \dots + \mu^{v\xi_j} \geq \mu^{j-\gamma}$  for  $1 \leq j \leq r$ . If  $\gamma \geq r - v - A$ , then*

$$\text{Vol}(\mathcal{T}_{\mu}(r, v, \gamma)) \ll_{\mu, A} \frac{1}{r!} \min \left\{ 1, \frac{(\gamma - r + v + A + 1)(\gamma + 1)}{r} \right\}.$$

*Proof.* If  $1 \leq r \leq 2v$ , then the result follows by Lemma 5.3 in [K10a] (see also Lemma 4.4 in [Fo08a]) and the trivial bound  $\text{Vol}(\mathcal{T}_\mu(r, v, \gamma)) \leq \text{Vol}(\Delta_r) = 1/r!$ . If  $r > 2v$ , then we have that  $\gamma \geq r - v - A \geq r/2 - A$  and, consequently,

$$\frac{(\gamma - r + v + A + 1)(\gamma + 1)}{r} \gg_A 1.$$

So the lemma holds in this case too by the trivial estimate  $\text{Vol}(\mathcal{T}_\mu(r, v, \gamma)) \leq 1/r!$ .  $\square$

*Proof of Lemma 5.3.* Let  $j \in \{1, \dots, k+1\}$  and  $\mathbf{r} \in \mathcal{R}_j$ . For each  $i \in \{1, \dots, k\}$ , let  $\mathcal{T}_i$  be the set of  $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) \in \Delta_{r_1} \times \dots \times \Delta_{r_k}$  such that

$$(5.11) \quad \min_{1 \leq s \leq k} \left\{ F_s(\boldsymbol{\xi}_s) \prod_{m=1}^{s-1} (k - m + 2)^{v_m - r_m} \right\} = \min_{1 \leq s \leq k} \{ F_s(\boldsymbol{\xi}_s) e^{-B_{1,s}} \} = F_i(\boldsymbol{\xi}_i) e^{-B_{1,i}}$$

and

$$F_s(\boldsymbol{\xi}_s) \leq C_k (k - s + 2)^{v_s - r_s} \quad (1 \leq s \leq k).$$

Then for every  $\boldsymbol{\xi} \in \mathcal{T}_i$  we have that

$$F_i(\boldsymbol{\xi}_i) e^{-B_{1,i}} \leq \min_{1 \leq s \leq k} \{ \min \{ C_k (k - s + 2)^{v_s - r_s}, 1 \} e^{-B_{1,s}} \},$$

which, together with (5.3), implies that

$$F_i(\boldsymbol{\xi}_i) \leq C_k e^{B_{1,i}} \min_{1 \leq s \leq k} e^{-\max\{B_{1,s}, B_{1,s+1}\}} = C_k e^{-\max\{B_{i,1}, \dots, B_{i,k+1}\}}.$$

Relation (5.3) and our assumption that  $\mathbf{r} \in \mathcal{R}_j$  imply that  $B_{i,j} = B_{i,s} + B_{s,j} \geq B_{i,s}$  for all  $s \in \{1, \dots, k+1\}$ , that is to say,  $\max\{B_{i,1}, \dots, B_{i,k+1}\} = B_{i,j}$  and, consequently,

$$F_i(\boldsymbol{\xi}_i) \leq C_k e^{-B_{i,j}}.$$

For  $i \in \{1, \dots, k\}$  and  $n \geq B_{i,j} \geq \max\{B_{i,i_0}, B_{i,i_0+1}, 0\}$ , let  $\mathcal{T}_i(n)$  be the set of  $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) \in \mathcal{T}_i$  such that

$$C_k e^{-n} < F_i(\boldsymbol{\xi}_i) \leq C_k e^{-n+1}.$$

Then for  $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k) \in \mathcal{T}_i(n)$  relations (5.3) and (5.11) imply that

$$F_{i_0}(\boldsymbol{\xi}_{i_0}) \geq e^{B_{i,i_0}} F_i(\boldsymbol{\xi}_i) > C_k e^{B_{i,i_0} - n}.$$

Hence, for every  $j \in \{1, \dots, r_{i_0}\}$ , we have that

$$\begin{aligned} \rho_{k-i_0+1}^{-j} \left( \rho_{k-i_0+1}^{v_{i_0} \xi_{i_0,1}} + \dots + \rho_{k-i_0+1}^{v_{i_0} \xi_{i_0,j}} \right) &\geq \max \left\{ (F_{i_0}(\boldsymbol{\xi}_{i_0}))^{1/(k-i_0+1)} - \rho_{k-i_0+1}^{-j}, \rho_{k-i_0+1}^{-j} \right\} \\ &\geq \frac{1}{2} (F_{i_0}(\boldsymbol{\xi}_{i_0}))^{1/(k-i_0+1)} \geq (\rho_{k-i_0+1})^{-\frac{n - B_{i,i_0}}{\log(k-i_0+2)}}, \end{aligned}$$

provided that  $C_k$  is large enough. So Lemma 5.5 gives us that

$$\begin{aligned}
U_{\mathbf{r}}^{(k+1)}(\mathbf{v}) &\leq \sum_{i=1}^k \int_{\tilde{\gamma}_i} e^{B_{i,1} F_i(\boldsymbol{\xi}_i)} d\boldsymbol{\xi} \\
&\leq C_k \sum_{i=1}^k \sum_{n \geq B_{i,j}} e^{B_{i,1}-n+1} \left( \prod_{\substack{1 \leq j \leq k \\ j \neq i_0}} \frac{1}{r_j!} \right) \text{Vol} \left( \mathcal{T}_{\rho_{k-i_0+1}} \left( r_{i_0}, v_{i_0}, \frac{n - B_{i,i_0}}{\log(k - i_0 + 2)} \right) \right) \\
&\ll_k \sum_{i=1}^k \frac{e^{B_{i,1}}}{r_1! \cdots r_k!} \sum_{n \geq B_{i,j}} \frac{1}{e^n} \min \left\{ 1, \frac{(n - B_{i,i_0} + 1)(n - B_{i,i_0+1} + 1)}{r_{i_0} + 1} \right\} \\
&\ll_k \sum_{i=1}^k \frac{e^{B_{i,1}}}{r_1! \cdots r_k!} \frac{1}{e^{B_{i,j}}} \min \left\{ 1, \frac{(B_{i,j} - B_{i,i_0} + 1)(B_{i,j} - B_{i,i_0+1} + 1)}{r_{i_0} + 1} \right\} \\
&= \frac{ke^{B_{j,1}}}{r_1! \cdots r_k!} \min \left\{ 1, \frac{(B_{i_0,j} + 1)(B_{i_0+1,j} + 1)}{r_{i_0} + 1} \right\},
\end{aligned}$$

which completes the proof of the lemma.  $\square$

We conclude this section with the proof of Lemma 5.4.

*Proof of Lemma 5.4.* Lemmas 5.2 and 5.3 imply that

$$\begin{aligned}
\sum_{\mathbf{r} \in (\mathbb{N} \cup \{0\})^k} S_{\mathbf{r}}^{(k+1)}(\mathbf{y}) &\ll_k \sum_{j=1}^{k+1} \sum_{\mathbf{r} \in \mathcal{R}_j} \left( \prod_{m=1}^{j-1} (k - m + 2)^{v_m} \right) \left( \prod_{m=j}^k (k - m + 2)^{r_m} \right) \\
(5.12) \quad &\times \min \left\{ 1, \frac{(1 + B_{i_0,j})(1 + B_{i_0+1,j})}{r_{i_0} + 1} \right\} \prod_{m=1}^k \frac{(v_m \log \rho_{k-m+1})^{r_m}}{r_m!} \\
&=: \sum_{j=1}^{k+1} T_j.
\end{aligned}$$

We fix  $j \in \{1, \dots, k+1\}$  and bound  $T_j$ . We have that  $\mathbf{r} \in \mathcal{R}_j$  if, and only if,

$$(5.13) \quad \sum_{m=i}^{j-1} \log(k - m + 2)(r_m - v_m) \geq 0 \quad (1 \leq i \leq j-1)$$

and

$$(5.14) \quad \sum_{m=j}^i \log(k - m + 2)(r_m - v_m) \leq 0 \quad (j \leq i \leq k).$$

Let  $\mathcal{R}_{1,j}$  be the set of vectors  $\mathbf{r}_1 = (r_1, \dots, r_{j-1}) \in (\mathbb{N} \cup \{0\})^{j-1}$  such that (5.13) holds and let  $\mathcal{R}_{2,j}$  be the set of vectors  $\mathbf{r}_2 = (r_j, \dots, r_k) \in (\mathbb{N} \cup \{0\})^{k-j+1}$  such that (5.14) holds. Note that if  $\mathbf{r}_1 \in \mathcal{R}_{1,j}$ , then

$$1 + B_{i_0,j} = 1 + B_{i_0,1} + B_{1,j} \leq (1 + \max\{0, B_{i_0,1}\})(1 + B_{1,j}) \ll_k (1 + \ell_1 + \dots + \ell_{i_0-1})(1 + B_{1,j}),$$

since (5.13) implies that  $B_{1,j} \geq 0$ . Similarly, if  $\mathbf{r}_2 \in \mathcal{R}_{2,j}$ , then

$$1 + B_{i_0+1,j} = 1 + B_{i_0+1,k+1} + B_{k+1,j} \ll_k (1 + r_{i_0+1} + \cdots + r_k)(1 + B_{k+1,j}),$$

since (5.14) implies that  $B_{k+1,j} \geq 0$ . So, if we set

$$\beta(\mathbf{r}) = \min \left\{ 1, \frac{(1 + \ell_1 + \cdots + \ell_{i_0-1})(1 + r_{i_0+1} + \cdots + r_k)}{r_{i_0} + 1} \right\},$$

then we have that

$$\begin{aligned} T_j &\ll_k \sum_{\substack{\mathbf{r}_i \in \mathcal{R}_{i,j} \\ i \in \{1,2\}}} \left( \prod_{m=1}^{j-1} (k - m + 2)^{v_m} \right) \left( \prod_{m=j}^k (k - m + 2)^{r_m} \right) \\ &\quad \times \beta(\mathbf{r})(1 + B_{1,j})(1 + B_{k+1,j}) \prod_{m=1}^k \frac{(v_m \log \rho_{k-m+1})^{r_m}}{r_m!}. \end{aligned}$$

For  $s \in \{0, 1, \dots, k\}$  set

$$\begin{aligned} T_{j,s} &= \sum_{\substack{\mathbf{r}_i \in \mathcal{R}_{i,j} \\ i \in \{1,2\}}} \left( \prod_{m=1}^{j-1} (k - m + 2)^{v_m} \right) \left( \prod_{m=j}^k (k - m + 2)^{r_m} \right) \\ &\quad \times (1 + B_{1,j})(1 + B_{k+1,j}) \frac{r_s + 1}{r_{i_0} + 1} \prod_{m=1}^k \frac{(v_m \log \rho_{k-m+1})^{r_m}}{r_m!}, \end{aligned}$$

where  $r_0 = 0$ . Then

$$(5.15) \quad T_j \ll_k \min \{T_{j,i_0}, (1 + \ell_1 + \cdots + \ell_{i_0-1})(T_{j,0} + T_{j,i_0+1} + T_{j,i_0+2} + \cdots + T_{j,k})\}.$$

Observe that  $T_{j,s}$  may be written as a product of two sums, with the first one ranging over  $\mathbf{r}_1 \in \mathcal{R}_{1,j}$  and the second one over  $\mathbf{r}_2 \in \mathcal{R}_{2,j}$ . Lemma 4.2(a) can be applied to the first of these sums (with  $j - 1$  in place of  $k$ ,  $\{v_i \log \rho_{k-i+1}\}_{i=1}^{j-1}$  in place of  $\{z_i\}_{i=1}^k$ ,  $\{\log(k - i + 2)\}_{i=1}^{j-1}$  in place of  $\{\lambda_i\}_{i=1}^k$  and  $\{k - i + 1\}_{i=1}^{j-1}$  in place of  $\{\mu_i\}_{i=1}^k$ ). Similarly, Lemma 4.2(b) can be applied to the second sum. As a result, we deduce that

$$(5.16) \quad T_{j,s} \ll_k \frac{1 + \ell_s}{1 + \ell_{i_0}} \left( \prod_{m=1}^k (k - m + 2)^{v_m} \right) \sum_{\substack{\mathbf{r}_i \in \mathcal{R}'_{i,j} \\ i \in \{1,2\}}} \prod_{m=1}^k \frac{(v_m \log \rho_{k-m+1})^{r_m}}{r_m!},$$

where  $\ell_0 = 0$ ,

$$\mathcal{R}'_{1,j} = \left\{ \mathbf{r}_1 \in (\mathbb{N} \cup \{0\})^{j-1} : -\log(k + 1) \leq \sum_{m=1}^{j-1} \log(k - m + 2)(r_m - v_m) \leq 0 \right\}$$

and

$$\mathcal{R}'_{2,j} = \left\{ \mathbf{r}_2 \in (\mathbb{N} \cup \{0\})^{k-j+1} : -\log(k + 1) \leq \sum_{m=j}^k \log(k - m + 2)(r_m - v_m) \leq 0 \right\}.$$

Clearly, we have that

$$\mathcal{R}'_{1,j} \times \mathcal{R}'_{2,j} \subset \left\{ \mathbf{r} \in (\mathbb{N} \cup \{0\})^k : -2 \log(k+1) \leq \sum_{m=1}^k \log(k-m+2)(r_m - v_m) \leq 0 \right\},$$

which, in combination with relation (5.16) and Lemmas 5.1 and 4.1(b), implies that

$$T_{j,s} \ll_k \frac{\ell_s + 1}{\ell_{i_0} + 1} \frac{1}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{k-i+2-Q((k-i+2)^\alpha)}.$$

By the above estimate and (5.15) we deduce that

$$T_j \ll_k \frac{\min \left\{ 1, \frac{(1 + \ell_1 + \dots + \ell_{i_0-1})(1 + \ell_{i_0+1} + \dots + \ell_k)}{\ell_{i_0}} \right\}}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{k-i+2-Q((k-i+2)^\alpha)}.$$

Finally, inserting this inequality and (2.10) into (5.12) proves the lemma.  $\square$

## 6. THE LOWER BOUND IN THEOREM 1.5: OUTLINE OF THE PROOF

As in the proof of the upper bound in Theorem 1.5, our starting point in order to prove the corresponding lower bound is Theorem 1.7. Also, we may assume that the numbers  $\ell_1, \dots, \ell_k$  are large enough, by Corollary 1.8. However, the arguments deviate significantly from those in Section 5. As in [Fo08a, Fo08b, K10a], our strategy is to construct a subset of  $\mathcal{P}_*^k(\mathbf{y})$  which contributes a positive proportion to  $S^{(k+1)}(\mathbf{y})$  and on which we have good control of the size of  $L^{(k+1)}(\mathbf{a})$  via Hölder's inequality. First, for  $P \in (1, +\infty)$  and  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$  set

$$W_{k+1}^P(\mathbf{a}) = \sum_{\substack{d_1 \dots d_i | a_1 \dots a_i \\ 1 \leq i \leq k}} \left( \sum_{\substack{d'_1 \dots d'_i | a_1 \dots a_i \\ |\log(d'_i/d_i)| < \log 2 \\ 1 \leq i \leq k}} 1 \right)^{P-1}.$$

We have the following inequality.

**Lemma 6.1.** *Let  $P \in (1, +\infty)$  and consider a finite set  $\mathcal{A} \subset \mathbb{N}^k$ . Then*

$$\left( \sum_{\mathbf{a} \in \mathcal{A}} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \dots a_k} \right)^{1/P} \left( \frac{1}{(\log 2)^k} \sum_{\mathbf{a} \in \mathcal{A}} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \dots a_k} \right)^{1-1/P} \geq \sum_{\mathbf{a} \in \mathcal{A}} \frac{\tau_{k+1}(\mathbf{a})}{a_1 \dots a_k}.$$

*Proof.* The proof is similar with the proof of Lemma 3.3 in [K10a]  $\square$

Our next goal is to bound

$$\sum_{\mathbf{a} \in \mathcal{A}} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \dots a_k}$$



from above for suitably chosen sets  $\mathcal{A} \subset \mathbb{N}^k$ . In order to construct these sets, recall the definition of the numbers  $\lambda_{i,j}$  and  $v_i$  and of the sets  $D_{i,j}$  from the beginning of Subsection 5.1. Then for  $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_k) \in (\mathbb{N} \cup \{0\})^{v_1} \times \dots \times (\mathbb{N} \cup \{0\})^{v_k}$  with  $\mathbf{g}_i = (g_{i,1}, \dots, g_{i,v_i})$  let

$$\mathcal{A}(\mathbf{g}) = \mathcal{A}_1(\mathbf{g}_1) \times \dots \times \mathcal{A}_k(\mathbf{g}_k),$$

where for each  $i \in \{1, \dots, k\}$   $\mathcal{A}_i(\mathbf{g}_i)$  is defined to be the set of square-free integers composed of exactly  $g_{i,j}$  prime factors from  $D_{i,j}$  for each  $j \in \{1, \dots, v_i\}$ . Set  $G_{i,0} = 0$  and  $G_{i,j} = g_{i,1} + \dots + g_{i,j}$ ,  $j = 1, \dots, v_i$ . We shall estimate

$$\sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k},$$

but first we need to introduce some new notation. Fix  $P \in (1, 2]$  and set

$$t_{i,j} = \frac{j + (k - i + 2 - j)^P}{k - i + 2} \quad (1 \leq i \leq k, 0 \leq j \leq k - i + 1).$$

Also, for integers  $1 \leq i \leq k$ ,  $\nu \geq 0$  and  $n \geq 0$  with  $\nu + n \leq k - i + 1$  and for  $\mathbf{g}_i \in (\mathbb{N} \cup \{0\})^{v_i}$ , set

$$T_i(\mathbf{g}_i; \nu, n) = \sum_{0=s_0 \leq s_1 \leq \dots \leq s_n \leq s_{n+1}=v_i} (\rho_{k-i+1}^{P-1})^{-(s_1+\dots+s_n)} \prod_{j=0}^n (t_{i,\nu+j})^{G_{i,s_{j+1}} - G_{i,s_j}}.$$

Lastly, we define

$$T(\mathbf{g}) = \sum_{\substack{0=J_0 \leq J_1 \leq \dots \leq J_k \leq k \\ J_i \geq i \ (1 \leq i \leq k)}} \prod_{i=1}^k (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} T_i(\mathbf{g}_i; J_{i-1} - i + 1, J_i - J_{i-1}).$$

**Lemma 6.2.** *Let  $\mathbf{r} \in \mathbb{N}^k$  and  $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_k) \in (\mathbb{N} \cup \{0\})^{v_1} \times \dots \times (\mathbb{N} \cup \{0\})^{v_k}$  such that  $G_{i,v_i} = r_i$  for all  $i \in \{1, \dots, k\}$ . Then*

$$\sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} \ll_k T(\mathbf{g}) \prod_{i=1}^k \frac{((k - i + 2) \log \rho_{i-1+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!}.$$

The proof of Lemma 6.2 will be given in Section 7. Next, we use the above result to show that  $W_{k+1}^P(\mathbf{a})$  is bounded on average over a union of suitable chosen sets  $\mathcal{A}(\mathbf{g})$ , which we construct below. Define

$$\mathcal{R}^* = \left\{ (r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : -\log(k+1) \leq \sum_{i=1}^k \log(k-i+2)(r_i - v_i) \leq 0, \right. \\ \left. |r_i - (k-i+2)^{\alpha \ell_i}| \leq \sqrt{\ell_i} \quad (1 \leq i \leq k) \right\}.$$

Fix  $\mathbf{r} \in \mathcal{R}^*$  and  $i \in \{1, \dots, k\}$  and set

$$u'_i = 1 + \frac{1}{\log(k-i+2)} \sum_{j=1}^{i-1} \log(k-j+2)(v_j - r_j)$$

and

$$w'_i = u'_i + v_i - r_i = 1 + \frac{1}{\log(k-i+2)} \sum_{j=1}^i \log(k-j+2)(v_j - r_j).$$

By Lemma 2.2 and the definition of  $i_0$  (see also the derivation of (2.6)), we have that

$$(6.1) \quad u'_i \asymp_k \begin{cases} 1 + \ell_1 + \cdots + \ell_{i-1} & \text{if } 1 \leq i \leq i_0, \\ 1 + \ell_i + \cdots + \ell_k & \text{if } i_0 + 1 \leq i \leq k, \end{cases}$$

and

$$(6.2) \quad w'_i \asymp_k \begin{cases} 1 + \ell_1 + \cdots + \ell_i & \text{if } 1 \leq i \leq i_0 - 1, \\ 1 + \ell_{i+1} + \cdots + \ell_k & \text{if } i_0 \leq i \leq k. \end{cases}$$

Define

$$u_i = \min \left\{ u'_i, \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \right\}$$

and

$$w_i = u_i + v_i - r_i = \min \left\{ w'_i, \frac{v_i - r_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \right\}.$$

Note that  $u_i \gg_k 1$  and  $w_i \gg_k 1$ , since  $r_i \asymp_k v_i$  for  $\mathbf{r} \in \mathcal{R}^*$ . Also, since

$$u'_i w'_i = (u'_i)^2 + (v_i - r_i)u'_i,$$

we have that  $u'_i w'_i \leq r_i$  exactly when

$$u'_i \leq \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2},$$

in which case  $u_i = u'_i$  and  $w_i = w'_i$ . On the other hand, if  $u'_i w'_i > r_i$ , then we find similarly that

$$u_i = \frac{r_i - v_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2} \quad \text{and} \quad w_i = \frac{v_i - r_i + \sqrt{(r_i - v_i)^2 + 4r_i}}{2}.$$

In any case, we have that

$$(6.3) \quad \beta_i := \frac{u_i w_i}{r_i} = \min \left\{ 1, \frac{u'_i w'_i}{r_i} \right\}.$$

Lastly, observe that

$$(6.4) \quad \beta_i \asymp_k \begin{cases} \beta & \text{if } i = i_0, \\ 1 & \text{else,} \end{cases}$$

by relations (6.1), (6.2) and (2.10). For every  $i \in \{2, \dots, k\}$  let  $\mathcal{G}_i(r_i)$  be the set of vectors  $\mathbf{g}_i \in (\mathbb{N} \cup \{0\})^{v_i}$  such that

$$(6.5) \quad G_{i,v_i} = r_i \quad \text{and} \quad G_{i,j} \leq j + u_i \quad (1 \leq j \leq v_i).$$

Also, let  $\mathcal{G}_1(r_1)$  be the set of vectors  $\mathbf{g}_1 = (g_{1,1}, \dots, g_{1,v_1}) \in (\mathbb{N} \cup \{0\})^{v_1}$  that satisfy (6.5) with  $i = 1$  and have the additional property that  $g_{1,j} = 0$  for  $1 \leq j \leq N - 1$ , where  $N = N(k)$

is a sufficiently large constant to be chosen later. Finally, let  $\mathcal{G}(\mathbf{r}) = \mathcal{G}_1(r_1) \times \cdots \times \mathcal{G}_k(r_k)$ . Then the following estimates hold.

**Lemma 6.3.** *For every  $\mathbf{r} \in \mathcal{R}^*$  we have that*

$$\sum_{\mathbf{g} \in \mathcal{G}(\mathbf{r})} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{1}{a_1 \cdots a_k} \gg_k \beta \prod_{i=1}^k \frac{\ell_i^{r_i}}{r_i!},$$

provided that  $N$  is large enough.

**Lemma 6.4.** *Assume that  $\alpha$  satisfies (1.1) for some fixed  $\epsilon > 0$ . If  $P = P(k, \epsilon)$  is close enough to 1, then for  $\mathbf{r} \in \mathcal{R}^*$  we have that*

$$\sum_{\mathbf{g} \in \mathcal{G}(\mathbf{r})} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} \ll_{k, \epsilon} \beta \prod_{i=1}^k \frac{((k-i+2)\ell_i)^{r_i}}{r_i!}.$$

Lemmas 6.3 and 6.4 will be proven in Section 8. Using these results, we complete the proof of Theorem 1.5.

*Proof of Theorem 1.5 (lower bound).* Assume that  $\alpha$  satisfies (1.1) for some fixed  $\epsilon > 0$ . Fix  $\mathbf{r} \in \mathcal{R}^*$ . For every  $\mathbf{a} \in \bigcup_{\mathbf{g} \in \mathcal{G}(\mathbf{r})} \mathcal{A}(\mathbf{g})$  we have that

$$\tau_{k+1}(\mathbf{a}) = \prod_{i=1}^k (k-i+2)^{r_i} \asymp_k \prod_{i=1}^k (k-i+2)^{v_i} \asymp_k \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{k-i+1},$$

by Lemma 5.1 and the definition of  $\mathcal{R}^*$ . Therefore

$$(6.6) \quad \sum_{\mathbf{g} \in \mathcal{G}(\mathbf{r})} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{L^{(k+1)}(\mathbf{a})}{a_1 \cdots a_k} \gg_{k, \epsilon} \beta \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{k-i+1} \prod_{i=1}^k \frac{\ell_i^{r_i}}{r_i!},$$

by Lemmas 6.1, 6.3 and 6.4. Also, relation (5.2) implies that

$$\bigcup_{\mathbf{r} \in \mathcal{R}^*} \bigcup_{\mathbf{g} \in \mathcal{G}(\mathbf{r})} \mathcal{A}(\mathbf{g}) \subset \mathcal{P}_*^k(\mathbf{y}).$$

Hence, combining (6.6) with Theorem 1.7, we deduce that

$$(6.7) \quad \frac{H^{(k+1)}(x, \mathbf{y}, 2\mathbf{y})}{x} \gg_{k, \epsilon} \beta e^{-(\ell_1 + \cdots + \ell_k)} \sum_{\mathbf{r} \in \mathcal{R}^*} \prod_{i=1}^k \frac{\ell_i^{r_i}}{r_i!}.$$

Finally, we have that

$$e^{-(\ell_1 + \cdots + \ell_k)} \sum_{\mathbf{r} \in \mathcal{R}^*} \prod_{i=1}^k \frac{\ell_i^{r_i}}{r_i!} \gg_k \frac{1}{\sqrt{\log \log y_k}} \prod_{i=1}^k \left( \frac{\log y_i}{\log y_{i-1}} \right)^{-Q((k-i+2)^\alpha)},$$

by Lemma 4.1(a), which completes the proof.  $\square$

## 7. THE METHOD OF LOW MOMENTS

This section is devoted to establishing Lemma 6.2. This will be done in three steps. Throughout this entire section we fix a vector  $\mathbf{r} \in \mathbb{N}^k$  and a vector  $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_k) \in (\mathbb{N} \cup \{0\})^{v_1} \times \dots \times (\mathbb{N} \cup \{0\})^{v_k}$  with  $G_{i,v_i} = r_i$  for all  $i \in \{1, \dots, k\}$ . Set  $R_i = \sum_{j=1}^i r_j$  and define

$$\mathcal{P}_{\mathbf{r}} = \{(Y_1, \dots, Y_k) : Y_i \subset \{1, \dots, R_i\}, Y_i \cap Y_j = \emptyset \text{ if } i \neq j\}.$$

Also, set

$$\mathcal{R}_i = \begin{cases} \{0, 1, \dots, R_1\} & \text{if } i = 1 \\ \{R_{i-1} + 1, \dots, R_i\} & \text{if } 2 \leq i \leq k. \end{cases}$$

For  $I \in \{0, 1, \dots, R_k\}$ , define  $E_{\mathbf{g}}(I) \in \bigcup_{i=1}^k \{0, 1, \dots, v_i\}$  as follows: If  $I = 0$ , set  $E_{\mathbf{g}}(I) = 0$ ; else, let  $i$  be the unique number in  $\{1, \dots, k\}$  such that  $R_{i-1} < I \leq R_i$  and define  $E_{\mathbf{g}}(I)$  by

$$G_{i,E_{\mathbf{g}}(I)-1} < I - R_{i-1} \leq G_{i,E_{\mathbf{g}}(I)}.$$

For  $\mathbf{Y} = (Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}$ ,  $\mathbf{m} = \{m_1, \dots, m_k\}$  a permutation of  $\{1, \dots, k\}$  and  $I_1, \dots, I_k \in \{0, 1, \dots, R_k\}$  put

$$M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m}) = \left\{ (Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} : \bigcup_{i=j}^k (Z_{m_i} \cap (I_j, R_k]) = \bigcup_{i=j}^k (Y_{m_i} \cap (I_j, R_k]) \ (1 \leq j \leq k) \right\}.$$

In addition, let

$$\mathcal{J} = \left\{ (\mathcal{J}_1, \dots, \mathcal{J}_k) : \mathcal{J}_i \subset \{1, \dots, k\}, \sum_{m=1}^i |\mathcal{J}_m| \geq i \ (1 \leq i \leq k), \mathcal{J}_i \cap \mathcal{J}_j = \emptyset \text{ if } i \neq j \right\}$$

and, for  $(\mathcal{J}_1, \dots, \mathcal{J}_k) \in \mathcal{J}$ , set  $J_i = |\mathcal{J}_1| + \dots + |\mathcal{J}_i| \geq i$  for all  $i \in \{0, \dots, k\}$ . Lastly, for a family of sets  $\{X_i\}_{i \in I}$  define

$$\mathcal{U}(\{X_i : i \in I\}) := \left\{ x \in \bigcup_{i \in I} X_i : |\{j \in I : x \in X_j\}| = 1 \right\}.$$

In particular,  $\mathcal{U}(Y, Z) = Y \Delta Z$  is the symmetric difference of  $Y$  and  $Z$ .

*Remark 7.1.* Assume that  $Y_1, \dots, Y_n$  and  $Z_1, \dots, Z_n$  satisfy  $Y_i \cap Y_j = Z_i \cap Z_j = \emptyset$  for  $i \neq j$ . Then

$$\mathcal{U}(\{Y_j \Delta Z_j : 1 \leq j \leq n\}) = \left( \bigcup_{j=1}^n Y_j \right) \Delta \left( \bigcup_{j=1}^n Z_j \right).$$

**7.1. Interpolating between  $L^1$  and  $L^2$  estimates.** The main difficulty in bounding  $W_{k+1}^P(\mathbf{a})$  when  $P \in (1, 2)$  is that it is hard to use combinatorial arguments directly due to the presence of the fractional exponent  $P - 1$  in the definition of  $W_{k+1}^P(\mathbf{a})$ . To overcome this difficulty, we perform a special type of interpolation between  $L^1$  and  $L^2$  estimates. This is accomplished in Lemma 7.1 below, which is a generalization of Lemma 3.5 in [K10a].

**Lemma 7.1.** *Let  $P \in (1, 2]$ ,  $\mathbf{r} \in (\mathbb{N} \cup \{0\})^k$  and  $\mathbf{g} = (g_1, \dots, g_k) \in (\mathbb{N} \cup \{0\})^{v_1} \times \dots \times (\mathbb{N} \cup \{0\})^{v_k}$  such that  $G_{i,v_i} = r_i$  for  $i = 1, \dots, k$ . Then*

$$\sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} \ll_k \sum_{(\mathcal{J}_1, \dots, \mathcal{J}_k) \in \mathcal{J}} \sum_{\mathbf{m}} \sum_{\substack{I_j \in \mathcal{R}_i \\ 1 \leq i \leq k, j \in \mathcal{J}_i}} \sum_{\mathbf{Y} \in \mathcal{P}_{\mathbf{r}}} (M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m}))^{P-1} \\ \times \prod_{i=1}^k \frac{(\log \rho_{k-i+1})^{r_i} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i}}{g_{i,1}! \cdots g_{i,v_i}!} \prod_{j \in \mathcal{J}_i} (\rho_{k-i+1}^{P-1})^{-E_{\mathbf{g}}(I_j)}.$$

*Proof.* Consider

$$(7.1) \quad \mathbf{a} = (a_1, \dots, a_k) = (p_1 \cdots p_{R_1}, p_{R_1+1} \cdots p_{R_2}, \dots, p_{R_{k-1}+1} \cdots p_{R_k}) \in \mathcal{A}(\mathbf{g})$$

such that

$$(7.2) \quad p_{R_{i-1}+G_{i,j-1}+1}, \dots, p_{R_{i-1}+G_{i,j}} \in D_{i,j} \quad (1 \leq i \leq k, 1 \leq j \leq v_i)$$

and the primes in each interval  $D_{i,j}$  for  $i = 1, \dots, k$  and  $j = 1, \dots, v_i$  are unordered. Since the number  $\prod_{i=1}^k a_i$  is square-free and  $\omega(a_i) = r_i$  for all  $i \in \{1, \dots, k\}$ , the  $k$ -tuples  $(d_1, \dots, d_k)$  with  $d_1 \cdots d_i | a_1 \cdots a_i$  for  $1 \leq i \leq k$  are in one to one correspondence with the  $k$ -tuples  $(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}$  via the relation

$$d_j = \prod_{i \in Y_j} p_i \quad (1 \leq j \leq k).$$

Using this observation twice, we find that

$$W_{k+1}^P(\mathbf{a}) = \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \left( \sum_{\substack{(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} \\ (7.3)}} 1 \right)^{P-1},$$

where for two  $k$ -tuples  $(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}$  and  $(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}}$  condition (7.3) is defined by

$$(7.3) \quad -\log 2 < \sum_{i \in Y_j} \log p_i - \sum_{i \in Z_j} \log p_i < \log 2 \quad (1 \leq j \leq k).$$

Moreover, each  $k$ -tuple  $(a_1, \dots, a_k) \in \mathcal{A}(\mathbf{g})$  has exactly  $\prod_{i,j} g_{i,j}!$  representations of the form given in (7.1), corresponding to all the possible permutations of the prime numbers

$p_1, \dots, p_{R_k}$  under condition (7.2). Hence

$$\begin{aligned} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} &= \left( \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq v_i}} \frac{1}{g_{i,j}!} \right) \sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2)}} \frac{1}{p_1 \cdots p_{R_k}} \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \left( \sum_{\substack{(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} \\ (7.3)}} 1 \right)^{P-1} \\ &= \left( \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq v_i}} \frac{1}{g_{i,j}!} \right) \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2)}} \frac{1}{p_1 \cdots p_{R_k}} \left( \sum_{\substack{(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} \\ (7.3)}} 1 \right)^{P-1}. \end{aligned}$$

So Hölder's inequality yields that

$$\begin{aligned} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} &\leq \left( \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq v_i}} \frac{1}{g_{i,j}!} \right) \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \left( \sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2)}} \frac{1}{p_1 \cdots p_{R_k}} \sum_{\substack{(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} \\ (7.3)}} 1 \right)^{P-1} \\ &\quad \times \left( \sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2)}} \frac{1}{p_1 \cdots p_{R_k}} \right)^{2-P}. \end{aligned}$$

Note that

$$\sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2)}} \frac{1}{p_1 \cdots p_{R_k}} \leq \prod_{i=1}^k \prod_{j=1}^{v_i} \left( \sum_{p \in D_{i,j}} \frac{1}{p} \right)^{g_{i,j}} \leq \prod_{i=1}^k (\log \rho_{k-i+1})^{r_i}$$

by (5.1) and, consequently,  
(7.4)

$$\sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} \leq \left( \prod_{i=1}^k \frac{(\log \rho_{k-i+1})^{(2-P)r_i}}{g_{i,1}! \cdots g_{i,v_i}!} \right) \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \left( \sum_{\substack{(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}} \\ (7.2), (7.3)}} \frac{1}{p_1 \cdots p_{R_k}} \right)^{P-1}.$$

Next, we estimate the sum over the primes above. In order to do so, we need to understand condition (7.3). Note that (7.3) is equivalent to

$$(7.5) \quad -\log 2 < \sum_{i \in Y_j \setminus Z_j} \log p_i - \sum_{i \in Z_j \setminus Y_j} \log p_i < \log 2 \quad (1 \leq j \leq k).$$

Fix two  $k$ -tuples  $(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}$  and  $(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}}$  and define the numbers  $I_1, \dots, I_k$  and  $m_1, \dots, m_k$  with  $I_i \in (Y_{m_i} \triangle Z_{m_i}) \cup \{0\}$  for all  $i \in \{1, \dots, k\}$  inductively, as follows (see the proof of [K10a, Lemma 3.5] for the motivation behind these definitions). Let

$$I_1 = \max \{ \mathcal{U}(Y_1 \triangle Z_1, \dots, Y_k \triangle Z_k) \cup \{0\} \}.$$

If  $I_1 = 0$ , set  $m_1 = k$ . Else, define  $m_1$  to be the unique element of  $\{1, \dots, k\}$  such that  $I_1 \in Y_{m_1} \triangle Z_{m_1}$ . Assume we have defined  $I_1, \dots, I_i$  for some  $i \in \{1, \dots, k-1\}$  with  $I_r \in (Y_{m_r} \triangle Z_{m_r}) \cup \{0\}$  for  $r = 1, \dots, i$ . Then set

$$I_{i+1} = \max \{ \mathcal{U}(\{Y_j \triangle Z_j : j \in \{1, \dots, k\} \setminus \{m_1, \dots, m_i\}\}) \cup \{0\} \}.$$

If  $I_{i+1} = 0$ , set  $m_{i+1} = \max\{\{1, \dots, k\} \setminus \{m_1, \dots, m_i\}\}$ . Otherwise, define  $m_{i+1}$  to be the unique element of  $\{1, \dots, k\} \setminus \{m_1, \dots, m_i\}$  such that  $I_{i+1} \in Y_{m_{i+1}} \triangle Z_{m_{i+1}}$ . This completes the inductive step.

Note that we must have  $\{m_1, \dots, m_k\} = \{1, \dots, k\}$ . Also, if we set

$$\mathcal{J}_i = \{1 \leq j \leq k : I_j \in \mathcal{R}_i\} \quad (1 \leq i \leq k),$$

then observe that  $(\mathcal{J}_1, \dots, \mathcal{J}_k) \in \mathcal{J}$ , since

$$J_i = \sum_{m=1}^i |\mathcal{J}_m| = |\{1 \leq j \leq k : I_j \leq R_i\}| \geq |\{1 \leq j \leq k : m_j \leq i\}| = i$$

for all  $i \in \{1, \dots, k\}$ . Set  $\mathcal{I} = \{I_j : 1 \leq j \leq k, I_j > 0\}$  and fix for the moment the primes  $p_i$  for  $i \in \{1, \dots, R_k\} \setminus \mathcal{I}$ . Then (7.5) becomes a system of linear inequalities with respect to the set of variables  $\{\log p_I : I \in \mathcal{I}\}$  that corresponds to a triangular matrix, up to a permutation of its rows. So a straightforward manipulation of the inequalities which constitute (7.5) implies that  $p_I \in [X_I, 4^k X_I]$  for  $I \in \mathcal{I}$ , where the numbers  $X_I$  depend only on the primes  $p_i$  for  $i \in \{1, \dots, R_k\} \setminus \mathcal{I}$  and the  $k$ -tuples  $(Y_1, \dots, Y_k)$  and  $(Z_1, \dots, Z_k)$ , which we have fixed. Consequently,

$$\sum_{\substack{p_I, \\ (7.2), (7.5)}} \prod_{I \in \mathcal{I}} \frac{1}{p_I} \ll_k \prod_{i=1}^k \prod_{\substack{j \in \mathcal{J}_i \\ I_j > 0}} \frac{1}{\log(\max\{\lambda_{i, E_g(I)-1}, X_{I_j}\})} \ll_k \prod_{i=1}^k \prod_{j \in \mathcal{J}_i} \frac{(\rho_{k-i+1})^{-E_g(I_j)}}{\log y_{i-1}},$$

by Lemma 5.1. So we find that

$$\sum_{\substack{p_1, \dots, p_{R_k} \\ (7.2), (7.5)}} \frac{1}{p_1 \cdots p_{R_k}} \ll_k \prod_{i=1}^k (\log \rho_{k-i+1})^{r_i} \prod_{j \in \mathcal{J}_i} \frac{(\rho_{k-i+1})^{-E_g(I_j)}}{\log y_{i-1}}$$

which, together with (7.4), implies that

$$\sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} \ll_k \left( \prod_{i=1}^k \frac{(\log \rho_{k-i+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!} \right) \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_r} \left( \sum_{(Z_1, \dots, Z_k) \in \mathcal{P}_r} \prod_{i=1}^k \prod_{j \in \mathcal{J}_i} \frac{(\rho_{k-i+1})^{-E_g(I_j)}}{\log y_{i-1}} \right)^{P-1}.$$

Note that

$$(7.7) \quad \prod_{i=1}^k (\log y_{i-1})^{|\mathcal{J}_i|} \asymp_k \prod_{i=1}^k e^{(k-J_i)\ell_i} \asymp_k \prod_{i=1}^k (\rho_{k-i+1})^{(k-J_i)v_i},$$

by Lemma 5.1. Moreover, the definition of the numbers  $I_1, \dots, I_k$  and  $m_1, \dots, m_k$  implies that

$$(I_j, R_k] \cap \mathcal{U}(\{Y_{m_r} \triangle Z_{m_r} : j \leq r \leq k\}) = \emptyset \quad (1 \leq j \leq k),$$

which is equivalent to

$$\bigcup_{r=j}^k (Z_{m_r} \cap (I_j, R_k]) = \bigcup_{r=j}^k (Y_{m_r} \cap (I_j, R_k]) \quad (1 \leq j \leq k),$$

by Remark 7.1. Hence for fixed  $(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}$ ,  $0 \leq I_1, \dots, I_k \leq R_k$  and  $\mathbf{m} = \{m_1, \dots, m_k\}$ , a permutation of  $\{1, \dots, k\}$ , the number of admissible  $k$ -tuples  $(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}}$  is at most  $M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m})$ . Combining this observation with (7.6) and (7.7) we deduce that

$$\begin{aligned} \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} &\ll_k \left( \prod_{i=1}^k \frac{(\log \rho_{k-i+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!} \right) \\ &\times \sum_{(Y_1, \dots, Y_k) \in \mathcal{P}_{\mathbf{r}}} \left( \sum_{I_1, \dots, I_k} \sum_{\mathbf{m}} M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m}) \prod_{\substack{1 \leq i \leq k \\ j \in \mathcal{J}_i}} (\rho_{k-i+1})^{-(k-J_i)v_i - E_{\mathbf{g}}(I_j)} \right)^{P-1}. \end{aligned}$$

Finally, the inequality  $(a+b)^{P-1} \leq a^{P-1} + b^{P-1}$  for  $a \geq 0$  and  $b \geq 0$ , which holds precisely when  $1 < P \leq 2$ , completes the proof of the lemma.  $\square$

**7.2. Combinatorial arguments.** In this subsection we use combinatorial arguments to calculate  $M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m})$  and, as a result, simplify the estimate given by Lemma 7.1. Note that the following lemma is similar to Lemma 3.6 in [K10a].

**Lemma 7.2.** *Let  $P \in (1, +\infty)$ ,  $\mathbf{r} \in (\mathbb{N} \cup \{0\})^k$ ,  $\mathbf{m} = \{m_1, \dots, m_k\}$  a permutation of  $\{1, \dots, k\}$ ,  $(\mathcal{J}_1, \dots, \mathcal{J}_k) \in \mathcal{J}$  and  $0 \leq I_1, \dots, I_k \leq R_k$  such that  $I_s \in \mathcal{R}_i$  for  $s \in \mathcal{J}_i$  and  $1 \leq i \leq k$ . Assume that  $\sigma \in S_k$  is a permutation such that  $I_{\sigma(1)} \leq \dots \leq I_{\sigma(k)}$ . Then*

$$\sum_{\mathbf{Y} \in \mathcal{P}_{\mathbf{r}}} (M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m}))^{P-1} \leq \prod_{i=1}^k \left( (k-i+2)^{r_i} \frac{(t_{i, J_i-i+1})^{R_i}}{(t_{i, J_{i-1}-i+1})^{R_{i-1}}} \prod_{J_{i-1} < j \leq J_i} \left( \frac{t_{i, j-i}}{t_{i, j-i+1}} \right)^{I_{\sigma(j)}} \right).$$

*Proof.* Set  $\sigma(0) = 0$ ,  $\sigma(k+1) = k+1$ ,  $I_0 = 0$  and  $I_{k+1} = R_k$ . First, we calculate  $M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m})$  for fixed  $\mathbf{Y} \in \mathcal{P}_{\mathbf{r}}$ . Let

$$\mathcal{N}_{i,j} = \mathcal{R}_i \cap (I_{\sigma(j)}, I_{\sigma(j+1)}] \quad (1 \leq i \leq k, J_{i-1} \leq j \leq J_i)$$

and

$$Y_{s,i,j} = Y_s \cap \mathcal{N}_{i,j}, \quad y_{s,i,j} = |Y_{s,i,j}| \quad (0 \leq s \leq k, 1 \leq i \leq k, J_{i-1} \leq j \leq J_i),$$

where

$$Y_0 = \{1, \dots, R_k\} \setminus \bigcup_{i=1}^k Y_i.$$

The  $k$ -tuple  $(Z_1, \dots, Z_k) \in \mathcal{P}_{\mathbf{r}}$  is counted by  $M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m})$  when

$$(7.8) \quad \bigcup_{s=j}^k (Z_{m_s} \cap (I_j, R_k]) = \bigcup_{s=j}^k (Y_{m_s} \cap (I_j, R_k]) \quad (1 \leq j \leq k).$$



So if we set

$$Z_{s,i,j} = Z_s \cap \mathcal{N}_{i,j} \quad (0 \leq s \leq k, \ 1 \leq i \leq k, \ J_{i-1} \leq j \leq J_i),$$

where

$$Z_0 = \{1, \dots, R_k\} \setminus \bigcup_{i=1}^k Z_i,$$

then (7.8) can be written as

$$(7.9) \quad \bigcup_{s=\sigma(t)}^k Z_{m_s,i,j} = \bigcup_{s=\sigma(t)}^k Y_{m_s,i,j} \quad (1 \leq i \leq k, \ J_{i-1} \leq j \leq J_i, \ 0 \leq t \leq j).$$

For  $j \geq 0$  let

$$\chi_j : \{0, 1, \dots, j, j+1\} \rightarrow \{\sigma(0), \sigma(1), \dots, \sigma(j), \sigma(k+1)\}$$

be the bijection uniquely determined by the property that  $\chi_j(0) < \dots < \chi_j(j+1)$ . So the sequence  $\chi_j(0), \dots, \chi_j(j+1)$  is the sequence  $\sigma(0), \dots, \sigma(j), \sigma(k+1)$  ordered increasingly. In particular,  $\chi_j(0) = \sigma(0) = 0$  and  $\chi_j(j+1) = \sigma(k+1) = k+1$ . Note that  $Z_{m_s,i,j} = Y_{m_s,i,j} = \emptyset$  if  $1 \leq m_s < i$ , by the definition of  $\mathcal{P}_r$ . So if we set  $m_0 = 0$  and

$$A_{t,i,j} = \{\chi_j(t) \leq s < \chi_j(t+1) : m_s \geq i \text{ or } s = 0\} \quad (1 \leq i \leq k, \ J_{i-1} \leq j \leq J_i, \ 0 \leq t \leq j),$$

then (7.9) is equivalent to

$$(7.10) \quad \bigcup_{s \in A_{t,i,j}} Z_{m_s,i,j} = \bigcup_{s \in A_{t,i,j}} Y_{m_s,i,j} \quad (0 \leq t \leq j),$$

for all  $1 \leq i \leq k$  and  $J_{i-1} \leq j \leq J_i$ . For such a pair  $(i, j)$ , let  $M_{i,j}$  be the set of mutually disjoint  $(k-i+2)$ -tuples  $(Z_{0,i,j}, Z_{i,i,j}, Z_{i+1,i,j}, \dots, Z_{k,i,j})$  that satisfy (7.10). Then

$$(7.11) \quad M_r(\mathbf{Y}; \mathbf{I}; \mathbf{m}) = \prod_{\substack{1 \leq i \leq k \\ J_{i-1} \leq j \leq J_i}} M_{i,j}.$$

Moreover, it is immediate from the definition of  $M_{i,j}$  that

$$M_{i,j} = \prod_{t=0}^j |A_{t,i,j}|^{\sum_{s \in A_{t,i,j}} y_{m_s,i,j}}$$

(with the standard notational convention that  $0^0 = 1$ ). Let

$$(7.12) \quad W_{t,i,j} = \bigcup_{s \in A_{t,i,j}} Y_{m_s,i,j} \quad \text{and} \quad w_{t,i,j} = |W_{t,i,j}| \quad (1 \leq i \leq k, \ J_{i-1} \leq j \leq J_i, \ 0 \leq t \leq j).$$

With this notation, we have that

$$M_{i,j} = \prod_{t=0}^j |A_{t,i,j}|^{w_{t,i,j}}.$$

Inserting the above relation into (7.11), we deduce that

$$M_r(\mathbf{Y}; \mathbf{I}; \mathbf{m}) = \prod_{i=1}^k \prod_{j=J_{i-1}}^{J_i} \prod_{t=0}^j |A_{t,i,j}|^{w_{t,i,j}}.$$

Therefore

$$S := \sum_{\mathbf{Y} \in \mathcal{P}_{\mathbf{r}}} (M_{\mathbf{r}}(\mathbf{Y}; \mathbf{I}; \mathbf{m}))^{P-1} = \prod_{i=1}^k \prod_{j=J_{i-1}}^{J_i} \sum_{Y_{0,i,j}, Y_{i,i,j}, \dots, Y_{k,i,j}} \prod_{t=0}^j |A_{t,i,j}|^{(P-1)w_{t,i,j}}.$$

Next, for fixed  $i \in \{1, \dots, k\}$ ,  $j \in \{J_{i-1}, \dots, J_i\}$  and  $W_{0,i,j}, \dots, W_{j,i,j}$ , a partition of  $\mathcal{N}_{i,j}$ , the number of  $Y_{0,i,j}, Y_{i,i,j}, \dots, Y_{k,i,j}$  that satisfy (7.12) is equal to

$$\prod_{t=0}^j |A_{t,i,j}|^{w_{t,i,j}}.$$

Consequently,

$$(7.13) \quad S = \prod_{i=1}^k \prod_{j=J_{i-1}}^{J_i} \sum_{W_{0,i,j}, \dots, W_{j,i,j}} \prod_{t=0}^j |A_{t,i,j}|^{Pw_{t,i,j}} = \prod_{i=1}^k \prod_{j=J_{i-1}}^{J_i} (|A_{0,i,j}|^P + \dots + |A_{j,i,j}|^P)^{|\mathcal{N}_{i,j}|},$$

by the multinomial theorem. Fix  $1 \leq i \leq k$  and  $J_{i-1} \leq j \leq J_i$  and set

$$K_{i,j} = \{0 \leq t \leq j : |A_{t,i,j}| \geq 1\}.$$

We claim that

$$(7.14) \quad j - i + 2 \leq |K_{i,j}| \leq k - i + 2.$$

Indeed, we have that

$$\{0\} \cup \{1 \leq s \leq k : m_s \geq i\} = \bigcup_{t \in K_{i,j}} A_{t,i,j} \subset \bigcup_{t \in K_{i,j}} \{s \in \mathbb{Z} : \chi_j(t) \leq s < \chi_j(t+1)\}.$$

The above relation implies that

$$k - i + 2 = |\{0\} \cup \{1 \leq s \leq k : m_s \geq i\}| = \sum_{t \in K_{i,j}} |A_{t,i,j}| \geq |K_{i,j}|$$

and

$$\begin{aligned} k - i + 2 &\leq \left| \bigcup_{t \in K_{i,j}} \{s \in \mathbb{Z} : \chi_j(t) \leq s < \chi_j(t+1)\} \right| \\ &= k + 1 - \left| \bigcup_{t \in \{0,1,\dots,j\} \setminus K_{i,j}} \{s \in \mathbb{Z} : \chi_j(t) \leq s < \chi_j(t+1)\} \right| \leq k - j + |K_{i,j}|, \end{aligned}$$

which together prove (7.14). Lastly, note that for  $n \leq X$  we have

$$\max \left\{ \sum_{j=1}^n x_j^P : \sum_{j=1}^n x_j = X, x_j \geq 1 \ (1 \leq j \leq n) \right\} = n - 1 + (X - n + 1)^P,$$

since the maximum of a convex function in a simplex occurs at its vertices. Therefore

$$\begin{aligned} |A_{0,i,j}|^P + \dots + |A_{j,i,j}|^P &\leq |K_{i,j}| - 1 + (k - i + 3 - |K_{i,j}|)^P \leq j - i + 1 + (k - j + 1)^P \\ &= (k - i + 2)t_{i,j-i+1}, \end{aligned}$$

by (7.14). Finally, inserting the above inequality into (7.13) yields

$$\begin{aligned} S &\leq \prod_{i=1}^k (k-i+2)^{r_i} \prod_{j=J_{i-1}}^{J_i} (t_{i,j-i+1})^{|\mathcal{N}_{i,j}|} \\ &= \prod_{i=1}^k (k-i+2)^{r_i} (t_{i,J_{i-1}-i+1})^{I_{\sigma(J_{i-1}+1)}-R_{i-1}} \left( \prod_{j=J_{i-1}+1}^{J_i-1} (t_{i,j-i+1})^{I_{\sigma(j+1)}-I_{\sigma(j)}} \right) (t_{i,J_i-i+1})^{R_i-I_{\sigma(J_i)}}, \end{aligned}$$

which completes the proof of the lemma.  $\square$

**7.3. Proof of Lemma 6.2.** In this last subsection we combine the results of Subsections 7.1 and 7.2 to show Lemma 6.2.

*Proof of Lemma 6.2.* By Lemmas 7.1 and 7.2 we have that

$$\begin{aligned} (7.15) \quad \sum_{\mathbf{a} \in \mathcal{A}(\mathbf{g})} \frac{W_{k+1}^P(\mathbf{a})}{a_1 \cdots a_k} &\ll_k \sum_{\substack{0=J_0 \leq J_1 \leq \cdots \leq J_k \leq k \\ J_i \geq i \ (1 \leq i \leq k)}} \sum_{\substack{0 \leq I_1 \leq \cdots \leq I_k \leq R_k \\ I_j \in \mathcal{R}_i, \ J_{i-1} < j \leq J_i \\ 1 \leq i \leq k}} \prod_{i=1}^k \left( \frac{((k-i+2) \log \rho_{k-i+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!} \right. \\ &\quad \times \frac{(t_{i,J_i-i+1})^{R_i}}{(\rho_{k-i+1}^{P-1})^{(k-J_i)v_i} (t_{i,J_{i-1}-i+1})^{R_{i-1}}} \prod_{J_{i-1} < j \leq J_i} (\rho_{k-i+1}^{P-1})^{-E_{\mathbf{g}}(I_j)} \left( \frac{t_{i,j-i}}{t_{i,j-i+1}} \right)^{I_j} \Big). \end{aligned}$$

Write  $e_j = E_{\mathbf{g}}(I_j)$  for  $i \in \{1, \dots, k\}$  and note that

$$0 \leq e_{J_{i-1}+1} \leq \cdots \leq e_{J_i} \leq v_i \quad (1 \leq i \leq k).$$

Moreover, for  $1 \leq i \leq k$  and  $J_{i-1} < j \leq J_i$  we have

$$\sum_{I_j \in \mathcal{R}_i, E_{\mathbf{g}}(I_j)=e_j} \left( \frac{t_{i,j-i}}{t_{i,j-i+1}} \right)^{I_j} \leq \sum_{G_{i,e_j-1}+R_{i-1} \leq I_j \leq G_{i,e_j}+R_{i-1}} \left( \frac{t_{i,j-i}}{t_{i,j-i+1}} \right)^{I_j} \ll_{k,P} \left( \frac{t_{i,j-i}}{t_{i,j-i+1}} \right)^{G_{i,e_j}+R_{i-1}},$$

since  $t_{i,j-i} > t_{i,j-i+1}$ . Inserting the above inequality into (7.15) completes the proof.  $\square$

## 8. THE LOWER BOUND IN THEOREM 1.5: COMPLETION OF THE PROOF

In this section we complete the proof of Theorem 1.5 by showing Lemmas 6.3 and 6.4.

**8.1. Preliminaries.** We state here some inequalities we will need later. For  $0 < h \leq x$  set

$$F(x, h) = \frac{(x+1) \log(x+1) - (x-h+1) \log(x-h+1)}{h}.$$

We summarize some properties of  $F$  in the following lemma.

**Lemma 8.1.** (a) For  $0 < h \leq x$  we have

$$\frac{\partial F(x, h)}{\partial h} < 0, \quad \frac{\partial F(x, h)}{\partial x} > 0 \quad \text{and} \quad \frac{\partial(F(x, x))}{\partial x} > 0.$$

(b) For  $0 < h \leq x - 1$  we have

$$F(x, h) > F(x - h, 1).$$

*Proof.* (a) We have that

$$\frac{\partial F(x, h)}{\partial h} = \frac{1}{h^2} \left[ h + (x + 1) \log \left( 1 - \frac{h}{x + 1} \right) \right] < 0 \quad (0 < h \leq x).$$

Also,

$$\frac{\partial F(x, h)}{\partial x} = \frac{1}{h} \log \left( \frac{x + 1}{x - h + 1} \right) > 0 \quad (0 < h \leq x).$$

Finally,

$$\frac{\partial(F(x, x))}{\partial x} = \frac{x - \log(x + 1)}{x^2} > 0 \quad (x > 0).$$

(b) Fix  $x > 1$  and note that it suffices to show that

$$g(h) = (x + 1) \log(x + 1) - (h + 1)(x - h + 1) \log(x - h + 1) + h(x - h) \log(x - h) > 0$$

for  $0 < h \leq x - 1$ . Since  $g(0) = 0$ , it is enough to show that  $g'(h) > 0$ . We have that

$$g'(h) = 1 + (2h - x) \log \left( \frac{x - h + 1}{x - h} \right).$$

If  $h \geq x/2$ , then  $g'(h) \geq 1$ . If  $0 < h < x/2$ , then

$$g'(h) > 1 - \frac{x - 2h}{x - h} = \frac{h}{x - h} > 0.$$

In any case, we have that  $g'(h) > 0$ , which completes the proof of the lemma.  $\square$

Finally, we have the following lemma.

**Lemma 8.2.** *The sequence*

$$\left\{ 1 - \frac{1}{\log(n + 2)} \log \left( \frac{(n + 2) \log(n + 2) - \log 4}{n} \right) \right\}_{n \in \mathbb{N}}$$

*is strictly increasing.*

*Proof.* For  $x > 0$  set

$$g(x) = \frac{(x + 2) \log(x + 2) - \log 4}{x} \quad \text{and} \quad G(x) = 1 - \frac{\log(g(x))}{\log(x + 2)}.$$

First, we check numerically that  $G(1) < G(2) < \dots < G(14)$ . Next, we handle the larger terms of the sequence. We have

$$G'(x) = \frac{h(x)}{x(x + 2)[(x + 2) \log(x + 2) - \log 4] \log^2(x + 2)},$$

where

$$h(x) = x[(x + 2) \log(x + 2) - \log 4] \log(g(x)) - (x + 2) \log(x + 2)(x - 2 \log(x + 2) + \log 4).$$

Observe that for  $x \geq 14$  we have  $\log(g(x)) \geq \log \log(x+2) \geq 1$ . Consequently,

$$\begin{aligned} h(x) &\geq -x \log 4 + 2(x+2) \log^2(x+2) - (\log 4)(x+2) \log(x+2) \\ &\geq (-x + 3(x+2) \log(x+2)) \log 4 > 0 \end{aligned}$$

for all  $x \geq 14$ , that is  $G'(x) > 0$  for  $x \geq 14$  and the desired result follows.  $\square$

**8.2. Estimates from order statistics.** Throughout this subsection we fix a vector  $\mathbf{r} \in \mathcal{R}^*$ . Our goal is to bound the quantities  $T_i(\mathbf{g}_i; \nu, n)$ , which were defined in Section 6, on average. To achieve this, we appeal to certain estimates from probability theory proven by Ford in [Fo08c]. Recall that

$$\Delta_r = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r : 0 \leq \xi_1 \leq \dots \leq \xi_r \leq 1\}.$$

For  $r \in \mathbb{N}$ ,  $u > 0$  and  $v \geq 1$  set

$$\begin{aligned} Q_r(u, v) &= r! \text{Vol} \left( \left\{ \boldsymbol{\xi} \in \Delta_r : \xi_i \geq \frac{i-u}{v} \quad (1 \leq i \leq r) \right\} \right) \\ &= \mathbf{Prob} \left( \xi_i \geq \frac{i-u}{v} \quad (1 \leq i \leq r) \mid \boldsymbol{\xi} \in \Delta_r \right). \end{aligned}$$

Then we have the following estimate, which essentially follows from Theorem 1 in [Fo08c]. This estimate is stated in [Fo07] too without proof. For the sake of completeness, we supply the details of its proof.

**Lemma 8.3.** *Let  $r \in \mathbb{N}$ ,  $u \geq 1$  and  $v \geq 1$ . If  $w = u + v - r \geq 1$ , then*

$$Q_r(u, v) \asymp \min \left\{ 1, \frac{uw}{r} \right\}.$$

*Proof.* The desired upper bound follows immediately by Theorem 1 in [Fo08c] and the trivial bound  $Q_r(u, v) \leq 1$ . For the lower bound we distinguish several cases. First, assume that  $v \geq 2r$ . Then

$$Q_r(u, v) \geq Q_r(1, v) = \frac{1+v-r}{v} \left( 1 + \frac{1}{v} \right)^{r-1} \asymp 1 = \min \left\{ 1, \frac{uw}{r} \right\},$$

by [Fo08c, Lemma 2.1(i)]. Next, consider the case  $v \leq 2r$ . Set

$$u' = \min \left\{ u, \frac{r-v + \sqrt{(r-v)^2 + 4r}}{2} \right\} \geq \frac{1}{2}$$

and

$$w' = u' + v - r = \min \left\{ w, \frac{v-r + \sqrt{(r-v)^2 + 4r}}{2} \right\} \geq \frac{1}{2}$$

By a similar argument with the one leading to (6.3), we have that

$$\min \left\{ 1, \frac{uw}{r} \right\} = \frac{u'w'}{r}.$$

Fix some constant  $C$ . If  $u' \geq C$  and  $w' \geq C$ , then the lower bound follows by Theorem 1 in [Fo08c] applied to  $Q_r(u', v) \leq Q_r(u, v)$ , provided that  $C$  is large enough. If  $1/2 \leq u' \leq w'$  and  $u' \leq C$ , then  $r \leq v \leq 2r$  and thus

$$Q_r(u, v) \geq Q_r(1, v) = \frac{1+v-r}{v} \left(1 + \frac{1}{v}\right)^{r-1} \asymp_C \frac{u'+v-r}{r} \asymp_C \frac{u'w'}{r},$$

by [Fo08c, Lemma 2.1(i)]. Finally, if  $1/2 \leq w' \leq u'$  and  $w' \leq C$ , then  $v \leq r$  and thus

$$Q_r(u, v) \geq Q_r(1+r-v, v) \gg \frac{1+r-v}{r} \asymp_C \frac{w'+r-v}{r} \asymp_C \frac{u'w'}{r},$$

by [Fo08b, Lemma 11.1]. In any case, we obtain the desired result.  $\square$

For  $r, v \in \mathbb{N}$  and  $u \geq 0$  set

$$\mathcal{G}_r(u, v) = \{(g_1, \dots, g_v) \in (\mathbb{N} \cup \{0\})^v : g_1 + \dots + g_v = r, g_1 + \dots + g_i \leq i + u \ (1 \leq i \leq v)\}.$$

Then an equivalent formulation of Lemma 8.3 is the following result.

**Lemma 8.4.** *Let  $r \in \mathbb{N}$ ,  $v \in \mathbb{N}$  and  $u \geq 0$ . If  $w = u + v - r \geq 0$ , then*

$$\sum_{\mathbf{g} \in \mathcal{G}_r(u, v)} \frac{1}{g_1! \cdots g_v!} \asymp \frac{v^r}{r!} \min \left\{ 1, \frac{(u+1)(w+1)}{r} \right\}.$$

*Proof.* For every  $\mathbf{g} \in \mathcal{G}_r(u, v)$ , let  $R(\mathbf{g})$  be the set of  $\boldsymbol{\xi} \in \Delta_r$  such that, for any  $i \in \{1, \dots, v\}$ , exactly  $g_i$  of the numbers  $\xi_j$  lie in  $[(i-1)/v, i/v)$ . Then

$$(8.1) \quad \text{Vol}(R(\mathbf{g})) = \frac{1}{v^r} \frac{1}{g_1! \cdots g_v!}.$$

Also, we have that  $g_1 + \dots + g_i \leq i + u$  if, and only if,  $\xi_{i+\lfloor u+1 \rfloor} \geq \frac{i}{v}$ . Hence summing (8.1) over  $\mathbf{g} \in \mathcal{G}_r(u, v)$  and applying Lemma 8.3 completes the proof.  $\square$

**Lemma 8.5.** *Let  $\mathbf{r} \in \mathcal{R}^*$ . Consider integers  $1 \leq i \leq k$ ,  $\nu \geq 0$  and  $n \geq 1$  with  $\nu + n \leq k - i + 1$  and  $\eta \in (0, 1]$ . There exists a constant  $c'_k > 0$  such that:*

(a) *If*

$$\left| \frac{F(k-i+1-\nu, n)}{(k-i+2)^{1-\alpha}} - 1 \right| \geq \eta,$$

*$P \leq 1 + \eta/c'_k$  and  $v_i \geq (c'_k/\eta)^2$ , then*

$$\sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; \nu, n)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P,\eta} \beta_i \frac{v_i^{r_i}}{r_i!} \max_{j \in \{0, n\}} (\rho_{k-i+1}^{P-1})^{-(n-j)v_i} (t_{i,\nu+j})^{r_i}.$$

(b) *If  $P \leq 1 + 1/c'_k$  and  $v_i \geq c'_k$ , then*

$$\sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; \nu, n)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P} \beta_i \frac{v_i^{r_i} e^{O_k((P-1)^2 v_i)}}{r_i!} \max_{j \in \{0, n\}} (\rho_{k-i+1}^{P-1})^{-(n-j)v_i} (t_{i,\nu+j})^{r_i}.$$

*Proof.* We will treat both parts together for the most part. The proofs of parts (a) and (b) will deviate only in the end. Set

$$\begin{aligned} S &= \sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; \nu, n)}{g_{i,1}! \cdots g_{i,v_i}!} \\ &= \sum_{0=s_0 \leq s_1 \leq \cdots \leq s_n \leq s_{n+1}=v_i} \sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{1}{g_{i,1}! \cdots g_{i,v_i}!} \prod_{j=0}^n (t_{i,\nu+j})^{G_{i,s_{j+1}}-G_{i,s_j}}. \end{aligned}$$

Fix  $0 = s_0 \leq s_1 \leq \cdots \leq s_n \leq s_{n+1} = v_i$  and let  $m_1, \dots, m_{n+1}$  be non-negative integers with

$$M_j = m_1 + \cdots + m_j \leq s_j + u_i \quad (1 \leq j \leq n), \quad M_{n+1} = m_1 + \cdots + m_{n+1} = r_i.$$

Also, put  $M_0 = 0$ . Then we have that

$$\begin{aligned} \sum_{\substack{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i) \\ G_{i,s_j}=M_j \\ 1 \leq j \leq n}} \frac{1}{g_{i,1}! \cdots g_{i,v_i}!} &= \prod_{j=0}^n \sum_{\substack{(g_{i,s_j+1}, \dots, g_{i,s_{j+1}}) \\ \in \mathcal{G}_{m_{j+1}}(u_i + s_j - M_j, s_{j+1} - s_j)}} \frac{1}{g_{i,s_j+1}! \cdots g_{i,s_{j+1}}!} \\ &\ll \min \left\{ \frac{u_i(u_i + s_1 - m_1 + 1)}{m_1 + 1}, \frac{w_i(u_i + s_n - M_n + 1)}{m_{n+1} + 1} \right\} \prod_{j=0}^n \frac{(s_{j+1} - s_j)^{m_{j+1}}}{m_{j+1}!}, \end{aligned}$$

by Lemma 8.4 applied for  $j = 0$  and  $j = n$ . Also, note that

$$\begin{aligned} \frac{u_i(u_i + s_1 - m_1 + 1)}{m_1 + 1} &\leq \frac{u_i(w_i + r_i - m_1 + 1)}{m_1 + 1} \leq \frac{u_i(w_i + 1)(r_i - m_1 + 1)}{m_1 + 1} \\ &\ll \beta_i \frac{r_i(r_i - m_1 + 1)}{m_1 + 1} \end{aligned}$$

and, similarly,

$$\frac{w_i(u_i + s_n - M_n + 1)}{m_{n+1} + 1} \leq \frac{w_i(u_i + 1)(s_n + 1)}{m_{n+1} + 1} \ll \beta_i \frac{r_i(s_n + 1)}{m_{n+1} + 1}.$$

So

$$\begin{aligned} (8.2) \quad S &\ll \beta_i r_i \sum_{0=s_0 \leq s_1 \leq \cdots \leq s_n \leq s_{n+1}=v_i} (\rho_{k-i+1}^{P-1})^{-(s_1 + \cdots + s_n)} \\ &\times \sum_{m_1 + \cdots + m_{n+1} = r_i} \min \left\{ \frac{r_i - m_1 + 1}{m_1 + 1}, \frac{s_n + 1}{m_{n+1} + 1} \right\} \prod_{j=0}^n \frac{(t_{i,\nu+j}(s_{j+1} - s_j))^{m_{j+1}}}{m_{j+1}!} \end{aligned}$$

The inner sum in the right hand side of (8.2) satisfies the following two upper bounds: it is at most

$$\begin{aligned} &\sum_{m_1=0}^{r_i} \frac{r_i - m_1 + 1}{m_1 + 1} \frac{(t_{i,\nu} s_1)^{m_1}}{m_1!} \frac{\left( \sum_{j=1}^n t_{i,\nu+j}(s_{j+1} - s_j) \right)^{r_i - m_1}}{(r_i - m_1)!} \\ &\ll_k \frac{1}{r_i!} \frac{v_i - s_1 + 1}{s_1 + 1} \left( \sum_{j=0}^n t_{i,\nu+j}(s_{j+1} - s_j) \right)^{r_i} \end{aligned}$$

and, also, it is at most

$$(s_n + 1) \sum_{m_{n+1}=0}^{r_i} \frac{(t_{i,\nu+n}(s_{n+1} - s_n))^{m_{n+1}}}{(m_{n+1} + 1)!} \frac{\left(\sum_{j=0}^{n-1} t_{i,\nu+j}(s_{j+1} - s_j)\right)^{r_i - m_{n+1}}}{(r_i - m_{n+1})!} \\ \ll_k \frac{1}{r_i!} \frac{s_n + 1}{v_i - s_n + 1} \left(\sum_{j=0}^n t_{i,\nu+j}(s_{j+1} - s_j)\right)^{r_i}.$$

Consequently,

$$(8.3) \quad S \ll_k \beta_i \frac{v_i^{r_i+1}}{r_i!} \sum_{0 \leq s_1 \leq \dots \leq s_n \leq v_i} (\rho_{k-i+1}^{P-1})^{-(s_1 + \dots + s_n)} \min \left\{ \frac{v_i - s_1 + 1}{s_1 + 1}, \frac{s_n + 1}{v_i - s_n + 1} \right\} \\ \times \left( \sum_{j=1}^n (t_{i,\nu+j-1} - t_{i,\nu+j}) \frac{s_j}{v_i} + t_{i,\nu+n} \right)^{r_i} \\ = \beta_i \frac{v_i^{r_i+1}}{r_i!} \sum_{0 \leq s_1 \leq \dots \leq s_n \leq v_i} g(s_1, s_n) \exp\{G(s_1, \dots, s_n)\},$$

where for  $\mathbf{x} = (x_1, \dots, x_n) \in [0, +\infty)^n$  we have set

$$G(\mathbf{x}) = \log \left( (\rho_{k-i+1}^{P-1})^{-(x_1 + \dots + x_n)} \left( \sum_{j=1}^n (t_{i,\nu+j-1} - t_{i,\nu+j}) \frac{x_j}{v_i} + t_{i,\nu+n} \right)^{r_i} \right)$$

and for  $(x, y) \in [0, +\infty)^2$  we have set

$$g(x, y) = \min \left\{ \frac{v_i - x + 1}{x + 1}, \frac{y + 1}{v_i - y + 1} \right\}.$$

We claim that

$$(8.4) \quad S \ll_{k,P} \beta_i \frac{v_i^{r_i+1}}{r_i!} \sum_{0 \leq s \leq v_i} g(s, s) \exp\{G(s, \dots, s)\}.$$

To show (8.4) we will make extensive use of the following simple fact: if  $b : [m, m+1] \rightarrow \mathbb{R}$  is a differentiable function satisfying  $b'(x) \geq \delta > 0$  for all  $x \in (m, m+1)$ , where  $\delta$  is a fixed positive number, then

$$(8.5) \quad \frac{e^{b(m+1)}}{e^{b(m)}} \geq e^\delta,$$

by the Mean Value Theorem. Fix a small positive constant  $\eta_0 = \eta_0(k)$  to be chosen later and define  $J \in \{0, 1, \dots, n-1\}$  as follows. If

$$\frac{F(k-i+1-\nu, 1)}{(k-i+2)^{1-\alpha}} < 1 + \eta_0,$$

then set  $J = 0$ ; else, put

$$J = \max \left\{ 1 \leq j \leq n-1 : \frac{F(k-i+1-\nu, j)}{(k-i+2)^{1-\alpha}} \geq 1 + \eta_0 \right\}.$$



Observe that

$$t_{i,j} = 1 + \frac{(k-i+2-j) \log(k-i+2-j)}{k-i+2} (P-1) + O_k((P-1)^2) \quad (0 \leq j \leq k-i+1).$$

Therefore if  $1 \leq j \leq J$ , then Lemma 8.1(a) yields that

$$\begin{aligned} (8.6) \quad & \frac{\partial}{\partial x_j} (G(\underbrace{x_j, \dots, x_j}_{j \text{ times}}, x_{j+1}, x_{j+2}, \dots, x_n)) \\ &= -j(P-1) \log \rho_{k-i+1} + \frac{r_i(t_{i,\nu} - t_{i,\nu+j})}{(t_{i,\nu} - t_{i,\nu+j})x_j + \sum_{m=j+1}^n (t_{i,\nu+m-1} - t_{i,\nu+m})x_m + t_{i,\nu+n}v_i} \\ &= j(P-1) \log \rho_{k-i+1} \left( -1 + \frac{F(k-i+1-\nu, j)}{(k-i+2)^{1-\alpha}} + O_k \left( P-1 + v_i^{-1/2} \right) \right) \\ &\geq \frac{\eta_0(P-1)j \log \rho_{k-i+1}}{2} > 0 \end{aligned}$$

uniformly in  $0 \leq x_j \leq \dots \leq x_n \leq v_i$ , provided that  $c'_k$  is large enough. So if  $J \geq 1$  and we fix  $0 \leq s_2 \leq \dots \leq s_n$ , then we have that

$$\sum_{0 \leq s_1 \leq s_2} g(s_1, s_n) \exp\{G(s_1, \dots, s_n)\} \ll_{k,P} g(s_2, s_n) \exp\{G(s_2, s_2, s_3, \dots, s_n)\},$$

by (8.6) with  $j = 1$  and (8.5). Similarly, if  $J \geq 2$  and we fix  $0 \leq s_3 \leq \dots \leq s_n \leq v_i$ , then

$$\sum_{0 \leq s_2 \leq s_3} g(s_2, s_n) \exp\{G(s_2, s_2, s_3, \dots, s_n)\} \ll_{k,P} g(s_3, s_n) \exp\{G(s_3, s_3, s_3, s_4, \dots, s_n)\}.$$

Continuing in the above fashion, we deduce that

$$\begin{aligned} (8.7) \quad & \sum_{0 \leq s_1 \leq \dots \leq s_n \leq v_i} g(s_1, s_n) \exp\{G(s_1, \dots, s_n)\} \\ & \ll_{k,P} \sum_{0 \leq s_{J+1} \leq \dots \leq s_n \leq v_i} g(s_{J+1}, s_n) \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+1 \text{ times}}, s_{J+2}, \dots, s_n)\}, \end{aligned}$$

which also holds trivially if  $J = 0$ . If, now,  $J = n-1$ , then (8.4) follows immediately by (8.7). So assume that  $J < n-1$ . Then Lemma 8.1(b) implies that

$$F(k-i-\nu-J, 1) < F(k-i+1-\nu, J+1) \leq 1 + \eta_0$$

and hence

$$F(k-i+2-\nu-j, 1) \leq F(k-i-\nu-J, 1) \leq 1 - \eta_0 \quad (J+2 \leq j \leq n),$$

provided that  $2\eta_0 \leq F(k-i+1-\nu, J+1) - F(k-i-\nu-J, 1)$ . Consequently,

$$\begin{aligned} (8.8) \quad & \frac{\partial G}{\partial x_j}(\mathbf{x}) = (P-1) \log \rho_{k-i+1} \left( -1 + \frac{F(k-i+2-\nu-j, 1)}{(k-i+2)^{1-\alpha}} + O_k \left( P-1 + v_i^{-1/2} \right) \right) \\ & \leq -\frac{\eta_0(P-1) \log \rho_{k-i+1}}{2} < 0 \quad (J+2 \leq j \leq n) \end{aligned}$$

uniformly in  $0 \leq x_1 \leq \dots \leq x_n \leq v_i$ , provided that  $c'_k$  is large enough. Thus, if we fix  $s_{J+1} \geq 0$  and  $v_i \geq s_n \geq s_{n-1} \geq \dots \geq s_{J+3} \geq s_{J+1}$ , then we find that

$$\sum_{s_{J+1} \leq s_{J+2} \leq s_{J+3}} \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+1 \text{ times}}, s_{J+2}, \dots, s_n)\} \ll_{k,P} \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+2 \text{ times}}, s_{J+3}, \dots, s_n)\},$$

by (8.8) with  $j = J + 2$  and (8.5). Similarly, if we fix  $s_{J+1} \geq 0$  and  $v_i \geq s_n \geq s_{n-1} \geq \dots \geq s_{J+4} \geq s_{J+1}$ , then we have

$$\sum_{s_{J+1} \leq s_{J+3} \leq s_{J+4}} \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+2 \text{ times}}, s_{J+3}, \dots, s_n)\} \ll_{k,P} \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+3 \text{ times}}, s_{J+4}, \dots, s_n)\}.$$

Continuing in this fashion, we deduce that

$$\begin{aligned} & \sum_{0 \leq s_{J+1} \leq \dots \leq s_n \leq v_i} g(s_{J+1}, s_n) \exp\{G(\underbrace{s_{J+1}, \dots, s_{J+1}}_{J+1 \text{ times}}, s_{J+2}, \dots, s_n)\} \\ & \ll_{k,P} \sum_{0 \leq s_{J+1} \leq v_i} g(s_{J+1}, s_{J+1}) \exp\{G(s_{J+1}, \dots, s_{J+1})\} \end{aligned}$$

which, together with (8.7) and (8.3), proves (8.4) in this case too. Finally, we use (8.4) to prove parts (a) and (b).

(a) First, assume that

$$\frac{F(k-i+1-\nu, n)}{(k-i+2)^{1-\alpha}} \geq 1 + \eta.$$

Note that

$$\begin{aligned} \frac{\partial}{\partial x}(G(x, \dots, x)) &= n(P-1) \log \rho_{k-i+1} \left( -1 + \frac{F(k-i+1-\nu, n)}{(k-i+2)^{1-\alpha}} + O_k(P-1 + v_i^{-1/2}) \right) \\ &\geq \frac{\eta(P-1)n \log \rho_{k-i+1}}{2} > 0 \end{aligned}$$

uniformly in  $0 \leq x \leq v_i$ , if  $c'_k$  is large enough. Hence

$$\sum_{0 \leq s \leq v_i} g(s, s) \exp\{G(s, \dots, s)\} \ll_{k,\eta,P} g(v_i, v_i) \exp\{G(v_i, \dots, v_i)\} = \frac{\exp\{G(v_i, \dots, v_i)\}}{v_i + 1},$$

by (8.5), which together with (8.4) yields the desired result. Similarly, if

$$\frac{F(k-i+1-\nu, n)}{(k-i+2)^{1-\alpha}} \leq 1 - \eta,$$

then we find that

$$\frac{\partial}{\partial x}(G(x, \dots, x)) \leq -\frac{\eta(P-1)n \log \rho_{k-i+1}}{2} < 0,$$

uniformly in  $0 \leq x \leq v_i$ , and therefore

$$\sum_{0 \leq s \leq v_i} g(s, s) \exp\{G(s, \dots, s)\} \ll_{k,\eta,P} \frac{\exp\{G(0, \dots, 0)\}}{v_i}.$$

Inserting this estimate into (8.4) gives us the desired result in this case as well.

(b) By (8.4), we have that

$$(8.9) \quad S \ll_{k,P} \beta_i \frac{v_i^{r_i+2}}{r_i!} \max_{0 \leq s \leq v_i} e^{G(s, \dots, s)} \ll_{k,P} \beta_i \frac{e^{O_k((P-1)^2 v_i)} v_i^{r_i}}{r_i!} \max_{0 \leq s \leq v_i} e^{G(s, \dots, s)}.$$

Since  $t_{i,j} = 1 + O_k(P-1)$  for all  $j$ , we find that, for any  $0 \leq s \leq v_i$ , we have that

$$\begin{aligned} \log \left( (t_{i,\nu} - t_{i,\nu+n}) \frac{s}{v_i} + t_{i,\nu+n} \right) &= (t_{i,\nu} - t_{i,\nu+n}) \frac{s}{v_i} + t_{i,\nu+n} - 1 + O_k((P-1)^2) \\ &= \frac{s}{v_i} \log \left( \frac{t_{i,\nu}}{t_{i,\nu+n}} \right) + \log(t_{i,\nu+n}) + O_k((P-1)^2) \end{aligned}$$

and, consequently,

$$\begin{aligned} \max_{0 \leq s \leq v_i} G(s, \dots, s) &= \max\{r_i \log(t_{i,\nu+n}), -(P-1)v_i \log \rho_{k-i+1} + r_i \log(t_{i,\nu})\} \\ &\quad + O_k((P-1)^2 v_i). \end{aligned}$$

Inserting the above estimate into (8.9) completes the proof of the lemma.  $\square$

The proof of the next lemma uses some ideas from the proof of [Fo08b, Lemmas 4.8 and 11.1] and [K10a, Lemma 3.8].

**Lemma 8.6.** *Let  $\mathbf{r} \in \mathcal{R}^*$  and  $i \in \{1, \dots, k\}$ . There is a constant  $c_k'' > 0$  such that*

$$\sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; 0, k-i+1)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P} \beta_i \frac{v_i^{r_i}}{r_i!} \prod_{j=1}^{i-1} (k-j+2)^{(P-1)(v_j-r_j)},$$

provided that  $P \leq 1 + 1/c_k''$ .

*Proof.* Since  $t_{i,0} > \cdots > t_{i,k-i} > t_{i,k-i+1} = 1$ , we have that

$$T_i(\mathbf{g}_i; 0, k-i+1) = \sum_{0 \leq s_1 \leq \cdots \leq s_{k-i+1} \leq v_i} \prod_{j=1}^{k-i+1} (\rho_{k-i+1}^{P-1})^{-s_j} \left( \frac{t_{i,j-1}}{t_{i,j}} \right)^{G_{i,s_j}}.$$

Also,

$$\prod_{j=1}^{k-i+1} \left( \frac{t_{i,j-1}}{t_{i,j}} \right)^{G_{i,s_j}} \leq \left( \frac{t_{i,0}}{t_{i,1}} \right)^{G_{i,s_1}} \prod_{j=2}^{k-i+1} \left( \frac{t_{i,j-1}}{t_{i,j}} \right)^{s_j + u_i} = \left( \frac{t_{i,0}}{t_{i,1}} \right)^{G_{i,s_1}} (t_{i,1})^{s_1 + u_i} \prod_{j=2}^{k-i+1} (t_{i,j-1})^{s_j - s_{j-1}}.$$

Thus, by setting

$$\lambda = \frac{t_{i,0}}{t_{i,1}} = \frac{(\rho_{k-i+1}^{P-1})^{k-i+1}}{t_{i,1}},$$

$m_1 = s_1$  and  $m_j = s_j - s_{j-1}$  for  $j = 2, \dots, k-i+1$ , we deduce that

$$(8.10) \quad T_i(\mathbf{g}_i; 0, k-i+1) \leq (t_{i,1})^{u_i} \sum_{\substack{m_1 + \cdots + m_{k-i+1} \leq v_i \\ m_j \geq 0 \ (1 \leq j \leq k-i+1)}} \lambda^{G_{i,m_1} - m_1} \prod_{j=2}^{k-i+1} \left( \frac{t_{i,j-1}}{(\rho_{k-i+1}^{P-1})^{k-i+2-j}} \right)^{m_j}.$$

Note that

$$\begin{aligned} \log(t_{i,j-1}) &= (P-1) \frac{(k-i-j+3) \log(k-i-j+3)}{k-i+2} + O_k((P-1)^2) \\ &< (P-1)(k-i-j+2) \log \rho_{k-i+1} \quad (2 \leq j \leq k-i+1), \end{aligned}$$

provided that  $P-1$  is small enough, by Lemma 8.1(a). Combining the above relation with (8.10) and summing the resulting inequality over  $\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)$ , we find that

$$(8.11) \quad \sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; 0, k-i+1)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_{k,P} (t_{i,1})^{u_i} \sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{1}{g_{i,1}! \cdots g_{i,v_i}!} \sum_{m=0}^{v_i} \lambda^{G_{i,m}-m} =: (t_{i,1})^{u_i} T.$$

Next, we claim that

$$(8.12) \quad T \leq \frac{v_i^{r_i}}{1-1/\lambda} \int_{\mathcal{S}} \left( 1 + \sum_{j=1}^{r_i} \lambda^{j-v_i \xi_j} \right) d\boldsymbol{\xi},$$

where

$$\mathcal{S} = \left\{ \boldsymbol{\xi} \in \Delta_{r_i} : \xi_j \geq \frac{j - \lfloor u_i + 1 \rfloor}{v_i} \quad (1 \leq j \leq r_i) \right\}.$$

To see this, fix  $\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)$  and consider the set  $I(\mathbf{g}_i)$  of vectors  $\boldsymbol{\xi} \in \Delta_{r_i}$  such that

$$|\{1 \leq j \leq r_i : s-1 \leq v_i \xi_j < s\}| = g_{i,s} \quad (1 \leq s \leq v_i).$$

Notice that if  $\boldsymbol{\xi} \in I(\mathbf{g}_i)$ , then  $v_i \xi_{j+\lfloor u_i+1 \rfloor} \geq j$  for all  $j$ , that is  $\boldsymbol{\xi} \in \mathcal{S}$ . Moreover,

$$\begin{aligned} \frac{1}{1-1/\lambda} \sum_{j=1}^{r_i} \lambda^{j-v_i \xi_j} &\geq \frac{1}{1-1/\lambda} \sum_{s=1}^{v_i} \lambda^{-s} \sum_{j: v_i \xi_j \in [s-1, s)} \lambda^j \geq \sum_{s=1}^{v_i} \sum_{m=s}^{v_i} \lambda^{-m} \sum_{j: v_i \xi_j \in [s-1, s)} \lambda^j \\ &= \sum_{m=1}^{v_i} \lambda^{-m} \sum_{j: v_i \xi_j < m} \lambda^j \geq \sum_{\substack{1 \leq m \leq v_i \\ G_{i,m} > 0}} \lambda^{-m+G_{i,m}} \geq -\frac{1}{1-1/\lambda} + \sum_{m=0}^{v_i} \lambda^{-m+G_{i,m}}. \end{aligned}$$

Lastly, we have that

$$\text{Vol}(I(\mathbf{g}_i)) = \frac{1}{v_i^{r_i}} \frac{1}{g_{i,1} \cdots g_{i,v_i}}.$$

Combining the above remarks, (8.12) follows. To bound the integral in the right hand side of (8.12), we proceed as in the proof of Lemma 4.9 in [Fo08b]. The only difference is that we use Lemma 8.3 from this paper in place of [Fo08b, Lemma 11.1]. This method gives us

$$T \ll_{k,P} \beta_i \frac{v_i^{r_i}}{r_i!} \lambda^{u_i}.$$

By the above estimate, (8.11) and (8.12), we deduce that

$$\sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; 0, k-i+1)}{g_{i,1}! \cdots g_{i,v_i}!} \ll_k \beta_i \frac{v_i^{r_i}}{r_i!} (k-i+2)^{(P-1)u_i}.$$

To complete the proof of the lemma, recall that

$$u_i \leq 1 + \frac{1}{\log(k-i+2)} \sum_{j=1}^{i-1} \log(k-j+2)(v_j - r_j).$$

□

**8.3. Proof of Lemmas 6.3 and 6.4.** In this subsection we establish Lemmas 6.3 and 6.4 thus completing all the steps in the proof of the lower bound implicit in Theorem 1.5.

*Proof of Lemma 6.3.* Since  $g_{1,j} \leq G_{1,j} \leq j + u_1 \leq j + 1$  for all  $j \in \{1, \dots, v_1\}$ , we have that

$$\begin{aligned} \sum_{a_1 \in \mathcal{A}_1(\mathbf{g}_1)} \frac{1}{a_1} &= \prod_{j=N}^{v_1} \frac{1}{g_{1,j}!} \left( \sum_{p_1 \in D_{1,j}} \frac{1}{p_1} \sum_{\substack{p_2 \in D_{1,j} \\ p_2 \neq p_1}} \frac{1}{p_2} \cdots \sum_{\substack{p_{g_{1,j}} \in D_{1,j} \\ p_{g_{1,j}} \notin \{p_1, \dots, p_{g_{1,j}-1}\}}} \frac{1}{p_{g_{1,j}}} \right) \\ (8.13) \quad &\geq \frac{1}{g_{1,1}! \cdots g_{1,v_1}!} \prod_{j=N}^{v_1} \left( \log \rho_k - \frac{g_{1,j}}{\lambda_{1,j-1}} \right)^{g_{1,j}} \\ &\geq \prod_{i=1}^k \frac{(\log \rho_k)^{r_1}}{g_{1,1}! \cdots g_{1,v_1}!} \prod_{j=N}^{v_1} \left( 1 - \frac{j+1}{(\log \rho_k) \exp \left\{ \rho_k^{j-L_k-1} \right\}} \right)^{j+1} \\ &\geq \frac{1}{2} \frac{(\log \rho_k)^{r_1}}{g_{1,N}! \cdots g_{1,v_1}!}, \end{aligned}$$

by Lemma 5.1, provided that  $N$  is large. Similarly, if  $i \in \{2, \dots, k\}$ , then  $g_{i,j} \leq G_{i,j} \leq j + u_i \leq j + c \log \log y_{i-1}$  for some  $c = c(k)$ . Therefore

$$\begin{aligned} \sum_{a_i \in \mathcal{A}_i(\mathbf{g}_i)} \frac{1}{a_i} &\geq \frac{(\log \rho_{k-i+1})^{r_i}}{g_{i,1}! \cdots g_{i,v_i}!} \prod_{j=1}^{v_i} \left( 1 - \frac{j + c \log \log y_{i-1}}{(\log \rho_{k-i+1})(\log y_{i-1}) \exp \left\{ \rho_{k-i+1}^{j-L_k-1} \right\}} \right)^{j+c \log \log y_{i-1}} \\ (8.14) \quad &\geq \frac{1}{2} \prod_{i=1}^k \frac{(\log \rho_{k-i+1})^{r_i}}{g_{i,N}! \cdots g_{i,v_i}!}, \end{aligned}$$

provided that  $y_{i-1} \geq y_1 \geq C'_k$  is large enough. Combine (8.13) and (8.14) with Lemmas 5.1 and 8.4 and relation (6.4) to complete the proof. □

*Proof of Lemma 6.4.* Fix  $\mathbf{r} \in \mathcal{R}^*$ . In view of Lemmas 5.1 and 6.2, it suffices to show that

$$(8.15) \quad \prod_{i=1}^k (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} \sum_{\mathbf{g}_i \in \mathcal{G}_{\mathbf{r}_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; J_{i-1} - i + 1, J_i - J_{i-1})}{g_{i,1}! \cdots g_{i,v_i}!} \ll_k \beta \prod_{i=1}^k \frac{v_i^{r_i}}{r_i!}$$

for every choice of integers  $0 = J_0 \leq J_1 \leq \dots \leq J_k \leq k$  with  $J_i \geq i$  for all  $i \in \{1, \dots, k\}$ . So fix such a  $(k+1)$ -tuple  $(J_0, J_1, \dots, J_k)$  and set

$$T_i = (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} \sum_{\mathbf{g}_i \in \mathcal{G}_{r_i}(u_i, v_i)} \frac{T_i(\mathbf{g}_i; J_{i-1} - i + 1, J_i - J_{i-1})}{g_{i,1}! \cdots g_{i,v_i}!} \quad (1 \leq i \leq k).$$

Also, let

$$I = \min\{1 \leq i \leq k : J_i = k\}$$

(note that  $J_k = k$ , so  $I$  is well-defined). We claim that

$$(8.16) \quad T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!} \times \begin{cases} (k-i+2)^{(P-1)(r_i-v_i)} & \text{if } 1 \leq i < I, \\ \max \left\{ 1, (k-I+2)^{(P-1)(r_I-v_I)}, \prod_{j=1}^{I-1} (k-j+2)^{(P-1)(v_j-r_j)} \right\} & \text{if } i = I, \\ 1 & \text{if } I < i \leq k. \end{cases}$$

Note that if inequality (8.16) is indeed true, then

$$\prod_{i=1}^k T_i \ll_{k,\epsilon} \left( \prod_{i=1}^k \beta_i \frac{v_i^{r_i}}{r_i!} \right) \max_{m \in \{1, I, I+1\}} \prod_{j=1}^{m-1} (k-j+2)^{(P-1)(r_j-v_j)} \ll_k \beta \prod_{i=1}^k \frac{v_i^{r_i}}{r_i!},$$

by relations (6.1), (6.2) and (6.4), that is (8.15) holds. So establishing (8.16) will complete the proof of the lemma.

Before embarking on the proof of (8.16), we introduce some notation and prove an intermediate result. For  $i \in \{1, \dots, k\}$  define  $J'_i \in \{J_{i-1}, J_i\}$  by

$$(\rho_{k-i+1}^{P-1})^{-(J_i-J'_i)v_i} (t_{i,J'_i-i+1})^{r_i} = \max_{j \in \{J_{i-1}, J_i\}} (\rho_{k-i+1}^{P-1})^{-(J_i-j)v_i} (t_{i,j-i+1})^{r_i}.$$

We claim that if  $i \leq J'_i \leq k-1$ , then

$$(8.17) \quad T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!} (k-i+2)^{(P-1)(r_i-v_i)},$$

provided that  $P-1$  is small enough. Indeed, Lemmas 5.1 and 8.5(b) give us that

$$(8.18) \quad \begin{aligned} & T_i \cdot (k-i+2)^{(P-1)(v_i-r_i)} \\ & \ll_k \frac{e^{O_k((P-1)^2 v_i)} v_i^{r_i} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} (\rho_{k-i+1}^{P-1})^{-(J_i-J'_i)v_i} (t_{i,J'_i-i+1})^{r_i}}{r_i! (k-i+2)^{(P-1)(r_i-v_i)}} \\ & = \frac{e^{O_k((P-1)^2 v_i)} v_i^{r_i}}{r_i!} (\rho_{k-i+1}^{P-1})^{(J'_i-i+1)v_i} \left( \frac{t_{i,J'_i-i+1}}{(k-i+2)^{P-1}} \right)^{r_i} \\ & = \frac{v_i^{r_i}}{r_i!} \exp \left\{ (P-1)(J'_i-i+1) \left[ \ell_i - \frac{F(k-i+1, J'_i-i+1)}{k-i+2} r_i + O_k((P-1)\ell_i+1) \right] \right\} \\ & = \frac{v_i^{r_i}}{r_i!} \exp \left\{ (P-1)(J'_i-i+1)\ell_i \left[ 1 - \frac{F(k-i+1, J'_i-i+1)}{(k-i+2)^{1-\alpha}} + O_k(P-1+\ell_i^{-1/2}) \right] \right\}. \end{aligned}$$

For every  $i \in \{1, \dots, k-1\}$ , condition (1.1) and Lemma 8.2 imply that

$$\alpha \geq 1 + \epsilon - \frac{1}{\log(k-i+2)} \log \left( \frac{(k-i+2) \log(k-i+2) - 2 \log 2}{k-i} \right)$$

or, equivalently, that

$$(8.19) \quad (k-i+2)^{\alpha-1} F(k-i+1, k-i) \geq (k-i+2)^\epsilon.$$

So if  $i \leq J'_i \leq k-1$ , then

$$(8.20) \quad (k-i+2)^{\alpha-1} F(k-i+1, J'_i-i+1) \geq (k-i+2)^{\alpha-1} F(k-i+1, k-i) \geq (k-i+2)^\epsilon,$$

by Lemma 8.1(a). Inserting the above inequality into (8.18) proves (8.17).

We are now in position to show (8.16). First, if  $I < i \leq k$ , then  $J_i = J_{i-1} = k$ . So

$$T_i(\mathbf{g}_i; J_{i-1}-i+1, J_i-J_{i-1}) = T_i(\mathbf{g}_i; k-i+1, 0) = 1$$

for every  $\mathbf{g}_i \in \mathcal{G}_i(r_i)$  and (8.16) follows immediately by Lemma 8.4. Next, let  $1 \leq i < I$ . If  $J'_i \geq i$ , then (8.16) follows by (8.17), since we also have that  $J'_i \leq J_i \leq J_{I-1} \leq k-1$ . Assume now that  $J'_i = i-1$ , in which case  $J_{i-1} = i-1$ . Then

$$\begin{aligned} \frac{F(k-i+1-(J_{i-1}-i+1), J_i-J_{i-1})}{(k-i+2)^{1-\alpha}} &= \frac{F(k-i+1, J_i-i+1)}{(k-i+2)^{1-\alpha}} \geq \frac{F(k-i+1, k-i)}{(k-i+2)^{1-\alpha}} \\ &\geq (k-i+2)^\epsilon, \end{aligned}$$

by Lemma 8.1(a) and relation (8.20). Therefore Lemma 8.5(a) applied with  $\eta = (k-i+2)^\epsilon - 1 > 0$  implies that

$$T_i \ll_{k,\epsilon} \beta_i \frac{v_i^{r_i}}{r_i!} (\rho_{k-i+1}^{P-1})^{-(k-J_i)v_i} (\rho_{k-i+1}^{P-1})^{-(J_i-J'_i)v_i} (t_{i,J'_i-i+1})^{r_i} = \beta_i \frac{v_i^{r_i}}{r_i!} (k-i+2)^{(P-1)(r_i-v_i)},$$

that is (8.16) holds in this case too. Finally, we bound from above  $T_I$ . If  $I \leq J'_I \leq k-1$  or  $J_{I-1} = I-1$ , then (8.16) follows immediately by (8.17) and Lemma 8.6, respectively. So suppose that  $J'_I \in \{I-1, k\}$  and  $J_{I-1} \geq I$ , in which case we must have  $J'_I = J_I = k$ . We separate two cases. Set

$$\eta_1 = \frac{F(k-I+1, k-I+1) - F(k-I, k-I)}{2(k-I+2)^{1-\alpha}} > 0$$

and assume first that

$$\frac{F(k-I+1, J'_I-I+1)}{(k-I+2)^{1-\alpha}} = \frac{F(k-I+1, k-I+1)}{(k-I+2)^{1-\alpha}} \geq 1 + \eta_1.$$

Inserting the above inequality into (8.18) implies that

$$T_I \ll_k \beta_I \frac{v_I^{r_I}}{r_I!} (k-I+2)^{(P-1)(r_I-v_I)},$$

provided that  $P-1$  is small enough, thus proving (8.16) in this case. Finally, assume that

$$\frac{F(k-I+1, k-I+1)}{(k-I+2)^{1-\alpha}} \leq 1 + \eta_1.$$

Then

$$\frac{F(k - I + 1 - (J_{I-1} - I + 1), J_I - J_{I-1})}{(k - I + 2)^{1-\alpha}} = \frac{F(k - J_{I-1}, k - J_{I-1})}{(k - I + 2)^{1-\alpha}} \leq \frac{F(k - I, k - I)}{(k - I + 2)^{1-\alpha}} \leq 1 - \eta_1$$

which, together with Lemma 8.5(a), shows that

$$T_I \ll_{k,\epsilon} \beta_I \frac{v_I^{r_I}}{r_I!} (\rho_{k-I+1}^{P-1})^{-(k-J_I)v_I} (\rho_{k-I+1}^{P-1})^{-(J_I-J'_I)v_I} (t_{I,J'_I-I+1})^{r_I} = \beta_I \frac{v_I^{r_I}}{r_I!},$$

thus proving (8.16) in this last case too. This completes the proof of the lemma.  $\square$

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